Chapter 10 Review

No calculator unless says "calc" in top right corner of question

1) The coefficient of x^6 in the Taylor series expansion about $x=0$ for $f(x)=\sin\left(x^2\right)$ is

1993 BC 43 calc

(A) $-\frac{1}{6}$ (B) 0 (C) $\frac{1}{120}$ (D) $\frac{1}{6}$ (E) 1

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

$$\sin(x^2) = x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \ldots$$

$$\sin(x^2) = x^2 - \underbrace{\frac{x^6}{3!}}_{} + \frac{x^{10}}{5!}$$

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2) A series expansion of $\frac{\sin t}{t}$ is

1973 BC 16

(A) $1-\frac{t^2}{3!}+\frac{t^4}{5!}-\frac{t^6}{7!}+\cdots$

(B) $\frac{1}{t}-\frac{t}{2!}+\frac{t^3}{4!}-\frac{t^5}{6!}+\cdots$

(C) $1+\frac{t^2}{3!}+\frac{t^4}{5!}+\frac{t^6}{7!}+\cdots$

(D) $\frac{1}{t}+\frac{t}{2!}+\frac{t^3}{4!}+\frac{t^5}{6!}+\cdots$

(E) $t-\frac{t^3}{3!}+\frac{t^5}{5!}-\frac{t^7}{7!}+\cdots$

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \ldots$$

$$\frac{\sin t}{t} = 1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \frac{t^6}{7!} + \ldots$$

3) If $f(x) = x\sin(2x)$, which of the following is the Taylor series for f about $x = 0$?

2008 BC 23

(A) $x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!} + \cdots$

(B) $x - \frac{4x^3}{2!} + \frac{16x^5}{4!} - \frac{64x^7}{6!} + \cdots$

(C) $2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \frac{128x^7}{7!} + \cdots$

(D) $2x^2 - \frac{2x^4}{3!} + \frac{2x^6}{5!} - \frac{2x^8}{7!} + \cdots$

(E) $2x^2 - \frac{8x^4}{3!} + \frac{32x^6}{5!} - \frac{128x^8}{7!} + \cdots$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

$\sin 2x = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \cdots$

$x\sin 2x = 2x^2 - \frac{8x^4}{3!} + \frac{32x^6}{5!}$

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4) What is the approximation of the value of sin 1 obtained by using the fifth-degree Taylor polynomial about $x = 0$ for $\sin x$?

1998 BC 14

(A) $1 - \frac{1}{2} + \frac{1}{24}$

(B) $1 - \frac{1}{2} + \frac{1}{4}$

(C) $1 - \frac{1}{3} + \frac{1}{5}$

(D) $1 - \frac{1}{4} + \frac{1}{8}$

(E) $1 - \frac{1}{6} + \frac{1}{120}$

$P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$

$P_5(1) = 1 - \frac{1}{3!} + \frac{1}{5!}$

5) The graph of the function represented by the Maclaurin series 1998 BC 89 Calc

$1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\ldots+\frac{(-1)^n x^n}{n!}+\ldots$ intersects the graph of $y=x^3$ at $x=$

(A) 0.773 (B) 0.865 (C) 0.929 (D) 1.000 (E) 1.857

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} = e^{-x}$$

* use calc to find where $y=e^{-x}$ and $y=x^3$ intersects

6) What is the coefficient of x^2 in the Taylor series for $\frac{1}{(1+x)^2}$ about $x=0$? 2003 BC 28

$$f(0) + f'(0)x + \frac{f''(0)x^2}{2!}$$

$f(x) = (1+x)^{-2}$

(A) $\frac{1}{6}$

(B) $\frac{1}{3}$

(C) 1

(D) 3

(E) 6

$$f'(x) = -2(1+x)^{-3}$$

$$f''(x) = 6(1+x)^{-4}$$

$$f''(0) = 6(1+0)^{-4}$$

$$= 6$$

$$\frac{6x^2}{2!}$$

$$\boxed{3}x^2$$

7) Let f be the function given by $f(x)=\ln(3-x)$. The third-degree Taylor polynomial for f about $x=2$ is

1997 BC 17

(A) $-(x-2)+\frac{(x-2)^2}{2}-\frac{(x-2)^3}{3}$

(B) $-(x-2)-\frac{(x-2)^2}{2}-\frac{(x-2)^3}{3}$

(C) $(x-2)+(x-2)^2+(x-2)^3$

(D) $(x-2)+\frac{(x-2)^2}{2}+\frac{(x-2)^3}{3}$

(E) $(x-2)-\frac{(x-2)^2}{2}+\frac{(x-2)^3}{3}$

$f(x)=\ln(3-x)$ $\quad f(2)=\ln(3-2)=\ln 1=0$

$f'(x)=-\frac{1}{3-x}$ $\quad f'(2)=-\frac{1}{3-2}=-1$

$=-(3-x)^{-1}$

$f''(x)=-(3-x)^{-2}$ $\quad f''(2)=-(3-2)^{-2}$

$=-1$

$f'''(x)=-2(3-x)^{-3}$ $\quad f'''(x)=-2(3-2)^{-3}$

$=-2$

$f(2)+f'(2)(x-2)+\frac{f''(2)(x-2)^2}{2!}+\frac{f'''(2)(x-2)^3}{3!}$

$0+(-1)(x-2)+\frac{(-1)(x-2)^2}{2!}+\frac{(-2)(x-2)^3}{3!}$

$-(x-2)-\frac{(x-2)^2}{2}-\frac{(x-2)^3}{3}$

8) The Maclaurin series for $\frac{1}{1-x}$ is $\sum_{n=0}^{\infty}x^n$. Which of the following is a power series expansion for $\frac{x^2}{1-x^2}$?

2003 BC 11

(A) $1+x^2+x^4+x^6+x^8+\cdots$

(B) $x^2+x^3+x^4+x^5+\cdots$

(C) $x^2+2x^3+3x^4+4x^5+\cdots$

(D) $x^2+x^4+x^6+x^8+\cdots$

(E) $x^2-x^4+x^6-x^8+\cdots$

$a_1=x^2$ $\quad r=x^2$

$x^2+x^4+x^6+\ldots$

9) The Taylor series for $\ln x$, centered at $x=1$, is $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n}$. Let f be the function given by the sum of the first three nonzero terms of this series. The maximum value of $|\ln x - f(x)|$ for $0.3 \le x \le 1.7$ is

1998 BC 83
calc

(A) 0.030 (B) 0.039 (C) 0.145 (D) 0.153 (E) 0.529

$$|\ln x - f(x)| = \left|\ln x - \underbrace{\left(\frac{x-1}{1} - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}\right)}_{\text{first 3 nonzero terms}}\right|$$

max occurs at $x = 0.3$

$$\le \left|\ln(0.3) - \left(\frac{(0.3-1)}{1} - \frac{(0.3-1)^2}{2} + \frac{(0.3-1)^3}{3}\right)\right|$$

$$= |-0.145|$$

* $(0.3-1)^2 = (1.7-1)^2$
$|\ln 0.3| > |\ln 1.7|$

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10) If $\sum_{n=0}^{\infty} a_n x^n$ is a Taylor series that converges to $f(x)$ for all real x, then $f'(1) =$

1998 BC 27

(A) 0 (B) a_1 (C) $\sum_{n=0}^{\infty} a_n$ (D) $\sum_{n=1}^{\infty} na_n$ (E) $\sum_{n=1}^{\infty} na_n^{\,n-1}$

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$f'(x) = \sum_{n=0}^{\infty} a_n n x^{n-1} \qquad f'(1) = \sum_{n=0}^{\infty} a_n n$$

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11) The sum of the infinite geometric series $\frac{3}{2} + \frac{9}{16} + \frac{27}{128} + \frac{81}{1{,}024} + \ldots$ is

1997 BC 14

(A) 1.60 (B) 2.35 (C) 2.40 (D) 2.45 (E) 2.50

$a_1 = 3/2 \qquad r = 3/8$

$$\frac{3/2}{1 - 3/8} = \frac{3}{2} \cdot \frac{8}{5} = \frac{12}{5} = 2.4$$

12) What are all values of x for which the series $\sum_{n=1}^{\infty}\left(\frac{2}{x^2+1}\right)^n$ converges? 2008 BC 16

(A) $-1<x<1$

(B) $x>1$ only

(C) $x\geq 1$ only

(D) $x<-1$ and $x>1$ only

(E) $x\leq -1$ and $x\geq 1$

$$\lim_{n\to\infty}\left|\frac{\left(\frac{2}{x^2+1}\right)^{n+1}}{\left(\frac{2}{x^2+1}\right)^n}\right|$$

$$\lim_{n\to\infty}\left|\frac{2}{x^2+1}\right| = \left|\frac{2}{x^2+1}\right| < 1$$

$$-1 < \frac{2}{x^2+1} < 1$$

$$\Rightarrow x^2+1>2$$

$$x^2>1$$

$$|x|>1$$

* or geometric

$$r=\frac{2}{x^2+1}$$

$$|r|<1$$

endpoints

$x=-1$ $\sum_{n=1}^{\infty}\left(\frac{2}{(-1)^2+1}\right)^n$

$= \sum_{n=1}^{\infty} 1^n$

diverges by n^{th} term test

$x=1$ $\sum_{n=1}^{\infty}\left(\frac{2}{(1)^2+1}\right)^n$

$= \sum_{n=1}^{\infty} 1^n$

diverges by n^{th} term test

$x<-1$ and $x>1$

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13) What are all values of x for which the series $\sum_{n=1}^{\infty}\frac{(x-1)^n}{n}$ converges? 1985 BC 31

(A) $-1\leq x<1$ (B) $-1\leq x\leq 1$ (C) $0<x<2$ (D) $0\leq x<2$ (E) $0\leq x\leq 2$

$$\lim_{n\to\infty}\left|\frac{(x-1)^{n+1}}{n+1}\cdot\frac{n}{(x-1)^n}\right|$$

$$\lim_{n\to\infty}\left|\frac{(x-1)n}{n+1}\right| = |x-1|<1$$

$$-1<x-1<1$$

$$0<x<2$$

endpoints

$x=0$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$$

1) $\lim_{n\to\infty}\frac{1}{n}=0$ ✓

2) $\frac{1}{n+1}\leq\frac{1}{n}$ ✓

converges by alternating series test

$x=2$

$$\sum_{n=1}^{\infty}\frac{(1)^n}{n} = \sum_{n=1}^{\infty}\frac{1}{n}$$

diverges by p-series test

14) The complete interval of convergence of the series $\sum_{k=1}^{\infty} \frac{(x+1)^k}{k^2}$ is

1969 BC 45

(A) $0 < x < 2$ (B) $0 \le x \le 2$ (C) $-2 < x \le 0$

(D) $-2 \le x < 0$ (E) $-2 \le x \le 0$

$$\lim_{k\to\infty} \left| \frac{(x+1)^{k+1}}{(k+1)^2} \cdot \frac{k^2}{(x+1)^k} \right|$$

$$\lim_{k\to\infty} \left| \frac{k^2 (x+1)}{(k+1)^2} \right| = |x+1| < 1$$

$-1 < x+1 < 1$

$-2 < x < 0$

endpoints

$\boxed{x=-2}$ $\sum_{k=1}^{\infty} \frac{(-2+1)^k}{k^2} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$

1) $\lim_{k\to\infty} \frac{1}{k^2} = 0$ ✓

2) $\frac{1}{(k+1)^2} \le \frac{1}{k^2}$ ✓

converges by alternating series test

$\boxed{x=0}$ $\sum_{k=1}^{\infty} \frac{(1)^k}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k^2}$

converges by p-series test

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15) What are all values of x for which the series $\sum_{n=1}^{\infty} \frac{(x+2)^n}{\sqrt{n}}$ converges?

1998 BC 84 Calc

(A) $-3 < x < -1$ (B) $-3 \le x < -1$ (C) $-3 \le x \le -1$ (D) $-1 \le x < 1$ (E) $-1 \le x \le 1$

$$\lim_{n\to\infty} \left| \frac{(x+2)^{n+1}}{(n+1)^{1/2}} \cdot \frac{n^{1/2}}{(x+2)^n} \right|$$

$$\lim_{n\to\infty} \left| \frac{n^{1/2}(x+2)}{(n+1)^{1/2}} \right| = |x+2| < 1$$

$-1 < x+2 < 1$

$-3 < x < -1$

endpoints

$\boxed{x=-3}$ $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

1) $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ ✓

2) $\frac{1}{\sqrt{n+1}} \le \frac{1}{\sqrt{n}}$ ✓

converges by alternating series test

$\boxed{x=-1}$ $\sum_{n=1}^{\infty} \frac{(1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} 1/\sqrt{n}$

diverges by p-series

16) Which of the following series converge? 1988 BC 44

I. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2n+1}$ → 1) $\lim_{n\to\infty}\frac{1}{2n+1}=0$ ✓ 2) $\frac{1}{2n+3}\le\frac{1}{2n+1}$ ✓ converges by alternating series test

II. $\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{3}{2}\right)^n$ $\lim_{n\to\infty}\left|\frac{3^{n+1}}{2^{n+1}(n+1)}\cdot\frac{2^n n}{3^n}\right|$

III. $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ $\lim_{n\to\infty}\left|\frac{3n}{2(n+1)}\right|=\left|\frac{3}{2}\right|>1$ diverges by ratio test

(A) I only
(B) II only
(C) III only
(D) I and III only
(E) I, II, and III

$\lim_{b\to\infty}\int_2^b\frac{1}{x\ln x}\,dx = \lim_{b\to\infty}\int_2^b\frac{1}{u}\,du$ $\lim_{b\to\infty}(\ln|\ln b|-\ln|\ln 2|)$

$u=\ln x$ $= \lim_{b\to\infty}\ln|\ln x|\Big|_2^b$ ∞

$du=1/x$ diverges by integral test

17) Which of the following series are convergent? 1985 BC 14

I. $1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}+\ldots$ converges by p-series

II. $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}+\ldots$ diverges by p-series

III. $1-\frac{1}{3}+\frac{1}{3^2}-\ldots+\frac{(-1)^{n+1}}{3^{n-1}}+\ldots$ 1) $\lim_{n\to\infty}\frac{3}{3^n}=0$ ✓ 2) $\frac{3}{3^{n+1}}\le\frac{3}{3^n}$ ✓

converges by alternating term test

(A) I only (B) III only (C) I and III only (D) II and III only (E) I, II, and III

18) For what values of x does the series $1+2^x+3^x+4^x+\cdots+n^x+\cdots$ converge? 1969 BC 32

(A) No values of x (B) $x<-1$ (C) $x\ge-1$ (D) $x>-1$ (E) All values of x

19) Let f be a positive, continuous, decreasing function such that $a_n = f(n)$. If $\sum_{n=1}^{\infty} a_n$ converges to k, which of the following must be true?

2008 BC 79
Calc

(A) $\lim_{n\to\infty} a_n = k$

★ integral test ★

(B) $\int_1^n f(x)\,dx = k$

(C) $\int_1^\infty f(x)\,dx$ diverges.

(D) $\int_1^\infty f(x)\,dx$ converges.

(E) $\int_1^\infty f(x)\,dx = k$

20) Which of the following sequences converge?

★ sequence
list of #s

1997 BC 76
Calc

series ∞ #s
summed

I. $\left\{\frac{5n}{2n-1}\right\}$ $\lim_{n\to\infty} = \frac{5}{2}$

II. $\left\{\frac{e^n}{n}\right\}$ ∞

sequence converge $\Rightarrow$ $\lim_{n\to\infty}$ approaches a #

or factor out e^n

III. $\left\{\frac{e^n}{1+e^n}\right\} \approx \frac{e^n}{e^n}$ $\lim_{n\to\infty} \frac{e^n}{e^n} = 1$

(A) I only (B) II only (C) I and II only (D) I and III only (E) I, II, and III

21) For what integer k, $k > 1$, will both $\sum_{n=1}^{\infty} \frac{(-1)^{kn}}{n}$ and $\sum_{n=1}^{\infty} \left(\frac{k}{4}\right)^n$ converge?

1998 BC 76
Calc

(A) 6 (B) 5 (C) 4 (D) 3 (E) 2

$\sum_{n=1}^{\infty} \left(\frac{k}{4}\right)^n$ converges for $k = 1, 2, \text{or } 3$ b/c geometric $|r|<1$

$\boxed{k=2}$ $\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-series

$\boxed{k=3}$ $\sum_{n=1}^{\infty} \frac{(-1)^{3n}}{n}$ converges by alternating series test

22) What are all values of p for which $\int_1^{\infty} \frac{1}{x^{2p}} dx$ converges? 2003 BC 6

(A) $p < -1$

(B) $p > 0$ ☆ p-series test ☆

(C) $p > \frac{1}{2}$

(D) $p > 1$

(E) There are no values of p for which this integral converges.

23) Which of the following series diverge? 2003 BC 24

$I. \sum_{n=0}^{\infty} \left(\frac{\sin 2}{\pi}\right)^n$ geometric $r = \frac{\sin 2}{\pi} < 1$ converges

$II. \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$ diverges by p-series

$III. \sum_{n=1}^{\infty} \left(\frac{e^n}{e^n+1}\right)$

$\lim_{b\to\infty} \int_1^b \frac{e^x}{e^x+1} dx$ $u = e^x + 1$, $du = e^x dx$

(A) III only

(B) I and II only

(C) I and III only

(D) II and III only

(E) I, II, and III

$\lim_{b\to\infty} \int_1^b \frac{1}{u} du$

$\lim_{b\to\infty} \ln|e^x+1| \Big|_1^b$

$\lim_{b\to\infty} \left(\ln|e^b+1| - \ln|e+1|\right)$

∞

diverges by Integral Test

24) Which of the following series converges for all real numbers x? 2008 BC 12

(A) $\sum_{n=1}^{\infty} \frac{x^n}{n}$ $\longrightarrow$ $\lim_{n\to\infty} \left| \frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n} \right| = \lim_{n\to\infty} \left| \frac{x \cdot n}{n+1} \right| = |x| < 1$

(B) $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ $\rightarrow$ $\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{x^n} \right| = |x| < 1$

(C) $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$ $\longrightarrow$ $\lim_{n\to\infty} \left| \frac{x^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{x^n} \right| = |x| < 1$

(D) $\sum_{n=1}^{\infty} \frac{e^n x^n}{n!}$

$\rightarrow$ $\lim_{n\to\infty} \left| \frac{e^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{e^n x^n} \right| = \lim_{n\to\infty} \left| \frac{ex}{n+1} \right| = 0 < 1$ always

(E) $\sum_{n=1}^{\infty} \frac{n! x^n}{e^n}$

$\rightarrow$ $\lim_{n\to\infty} \left| \frac{(n+1)!\, x^{n+1}}{e^{n+1}} \cdot \frac{e^n}{n!\, x^n} \right|$

$= \lim_{n\to\infty} \left| \frac{(n+1)x}{e} \right| = \infty < 1$ never