6. The function f has a Taylor series about $x = 1$ that converges to $f(x)$ for all x in the interval of convergence. It is known that $f(1) = 1$, $f'(1) = -\frac{1}{2}$, and the nth derivative of f at $x = 1$ is given by $f^{(n)}(1) = (-1)^n \frac{(n-1)!}{2^n}$ for $n \geq 2$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

(b) The Taylor series for f about $x = 1$ has a radius of convergence of 2. Find the interval of convergence. Show the work that leads to your answer.

(c) The Taylor series for f about $x = 1$ can be used to represent $f(1.2)$ as an alternating series. Use the first three nonzero terms of the alternating series to approximate $f(1.2)$.

(d) Show that the approximation found in part (c) is within 0.001 of the exact value of $f(1.2)$.

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a) $P_3(x) = f(x) + f'(x)(x-1) + \frac{f''(x)(x-1)^2}{2!} + \frac{f'''(x)(x-1)^3}{3!}$

$= 1 - \frac{1}{2}(x-1) + \frac{1}{4}\frac{(x-1)^2}{2} - \frac{2}{8}\frac{(x-1)^3}{6}$

$= 1 - \frac{1}{2}(x-1) + \frac{1}{8}(x-1)^2 - \frac{1}{24}(x-1)^3$

b) $R = 2$ centered at $x = 1$

$-1 < x < 3$

Let $x = -1$

$1 + 1 + \frac{1}{8}(-1-1)^2 - \frac{1}{24}(-1-1)^3$

$\underbrace{1 + 1 + \frac{1}{2} + \frac{1}{3} + \ldots}$

harmonic series
↳ diverges

Let $x = 3$

$1 - 1 + \frac{1}{8}(3-1)^2 - \frac{1}{24}(3-1)^3$

$\underbrace{1 - 1 + \frac{1}{2} - \frac{1}{3}}$

alternating $\Sigma \, 1/n \rightarrow$ converges

Therefore the interval of convergence is $-1 < x \leq 3$

c) $f(1.2) \approx 1 - \frac{1}{2}(1.2-1) + \frac{1}{8}(1.2-1)^2$

$= 1 - \frac{1}{2}(0.2) + \frac{1}{8}(0.2)^2$

$= 0.905$

d) $\left| f(1.2) - P_2(x) \right| = \left| f(1.2) - 0.905 \right| \leq \text{error}$

$$= \left| \frac{f^{(3)}(x)\,(x-1)^3}{3!} \right|$$

$$= \left| \frac{f^{(3)}(x)\,(1.2-1)^3}{3!} \right|$$

$$= \left| \frac{-2}{8} \frac{(0.2)^3}{3!} \right|$$

$$= \frac{1}{3000} \leq 0.001$$

6. Let $f(x) = \ln(1 + x^3)$.

(a) The Maclaurin series for $\ln(1 + x)$ is $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1} \cdot \frac{x^n}{n} + \cdots$. Use the series to write the first four nonzero terms and the general term of the Maclaurin series for f.

(b) The radius of convergence of the Maclaurin series for f is 1. Determine the interval of convergence. Show the work that leads to your answer.

(c) Write the first four nonzero terms of the Maclaurin series for $f'(t^2)$. If $g(x) = \int_0^x f'(t^2)\, dt$, use the first two nonzero terms of the Maclaurin series for g to approximate $g(1)$.

(d) The Maclaurin series for g, evaluated at $x = 1$, is a convergent alternating series with individual terms that decrease in absolute value to 0. Show that your approximation in part (c) must differ from $g(1)$ by less than $\frac{1}{5}$.

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a) $f(x) = \ln(1+x^3) = x^3 - \frac{(x^3)^2}{2} + \frac{(x^3)^3}{3} - \frac{(x^3)^4}{4} + \ldots + (-1)^{n+1} \frac{(x^3)^n}{n} + \ldots$

$= x^3 - \frac{x^6}{2} + \frac{x^9}{3} - \frac{x^{12}}{4} + \ldots + (-1)^{n+1} \frac{x^{3n}}{n} + \ldots$

b) let $x = -1$

$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-1)^{3n}}{n}$

$= \sum_{n=1}^{\infty} \frac{(-1)^{4n+1}}{n}$

$= \sum_{n=1}^{\infty} \frac{(-1)}{n}$

diverges b/c harmonic series (or $p=1$)

let $x = 1$

$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(1)^{3n}}{n}$

$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

converges by alternating term test

Therefore interval of convergence is

$-1 < x \leq 1$

(t^2)

$$) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{3n\,x^{3n-1}}{n}$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1}\, 3x^{3n-1} = 3x^2 - 3x^5 + 3x^8 - 3x^{11} + \ldots$$

$$) = \sum_{n=1}^{\infty} (-1)^{n+1}\, 3\,(t^2)^{3n-1}$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1}\, 3t^{6n-2} = 3t^4 - 3t^{10} + 3t^{16} - 3t^{22} + \ldots$$

$$) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{3t^{6n-1}}{6n-1} = \frac{3x^5}{5} - \frac{3x^{11}}{11} + \frac{3x^{17}}{17} - \frac{3x^{23}}{23} + \ldots$$

$$) \approx \frac{3}{5} - \frac{3}{11} = \frac{18}{55}$$

$$- P_2(x)\Big| = \text{error}$$

$$- \frac{18}{55}\Big| \leq a_{n+1}$$

$$= a_3$$

$$= \frac{3(1)^{17}}{17}$$

$$= \frac{3}{17} < \frac{1}{5}$$