REVIEW & INSTRUCTIONAL PACKET

Interval Notation

Sets of real numbers, such as those graphed in Example 4, can be conveniently described using **interval notation.** For example, instead of writing $\{x \mid -\frac{3}{2} \le x < 3\}$, we can simply write $[-\frac{3}{2}, 3)$. Notice that we are following our graphing convention of using a parenthesis to indicate that an endpoint of the interval is not included and a bracket to indicate that an endpoint is included. The set $\{x \mid x \ge 3\}$ can be written as $[3, \infty)$. Note that the symbol "∞" (infinity) does not represent a real number but rather indicates that the interval includes all points to the right of 3. Table 2 summarizes the nine types of intervals.

or open circle

solid circle or

TABLE 2 Interval notation

Interval	Set-builder notation	Graph
(a, b)	$\{x \mid a < x < b\}$	a b
$[a, b]$	$\{x \mid a \le x \le b\}$	a b
$(a, b]$	$\{x \mid a < x \le b\}$	a b
$[a, b)$	$\{x \mid a \le x < b\}$	a b
(a, ∞)	$\{x \mid x > a\}$	a
$[a, \infty)$	$\{x \mid x \ge a\}$	a
$(-\infty, b)$	$\{x \mid x < b\}$	b
$(-\infty, b]$	$\{x \mid x \le b\}$	b
$(-\infty, \infty)$	$\{x \mid -\infty < x < \infty\}$	

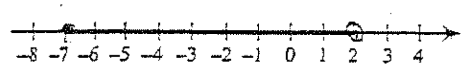
EXAMPLE 5 **Using Interval Notation**

Write the given set using interval notation and sketch its graph.

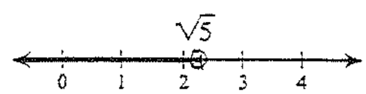
a. $\{x \mid -7 \le x < 2\}$ b. $\{s \mid s < \sqrt{5}\}$

SOLUTION

a. $[-7, 2)$



b. $(-\infty, \sqrt{5})$



EXAMPLE 6 **Converting from Interval Notation to Set-Builder Notation**

Express each of the following sets using set-builder notation.

a. $(2, 5]$ b. $(-\infty, 3)$ c. $[-4, \infty)$

SOLUTION

a. $\{x \mid 2 < x \le 5\}$ b. $\{x \mid x < 3\}$ c. $\{x \mid x \ge -4\}$

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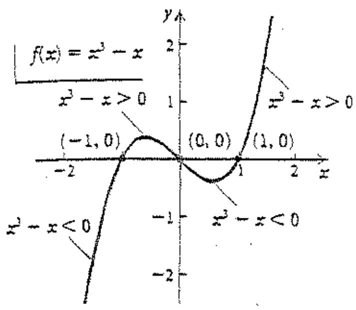
Polynomial and Rational Inequalities

3.5

Bittinger, PreCalculus



X	Y1
−2	−6
−.5	.375
.5	−.375
2	6

X =

> To solve a polynomial inequality:
>
> 1. Find an equivalent inequality with 0 on one side.
> 2. Solve the related polynomial equation.
> 3. Use the solutions to divide the x-axis into intervals. Then select a test value from each interval and determine the polynomial's sign on the interval.
> 4. Determine the intervals for which the inequality is satisfied and write interval notation or set-builder notation for the solution set. Include the endpoints of the intervals in the solution set if the inequality symbol is $\le$ or $\ge$.

- *Solve polynomial and rational inequalities.*

We will use a combination of algebraic and graphical methods to solve polynomial and rational inequalities.

Polynomial Inequalities

Just as a quadratic equation can be written in the form $ax^2 + bx + c = 0$, a **quadratic inequality** can be written in the form $ax^2 + bx + c \;\square\; 0$, where $\square$ is $<$, $>$, $\le$, or $\ge$. Here are some examples of quadratic inequalities:

$$3x^2 - 2x - 5 > 0, \qquad -\tfrac{1}{2}x^2 + 4x - 7 \le 0.$$

Quadratic inequalities are one type of **polynomial inequality**. Other examples of polynomial inequalities are

$$-2x^4 + x^2 - 3 < 7, \qquad \tfrac{2}{3}x + 4 \ge 0, \quad \text{and} \quad 4x^3 - 2x^2 > 5x + 7.$$

When the inequality symbol in a polynomial inequality is replaced with an equals sign, a **related equation** is formed. Polynomial inequalities can be easily solved once the related equation has been solved.

EXAMPLE 1 Solve: $x^3 - x > 0$.

Solution We are asked to find all x-values for which $x^3 - x > 0$. To locate these values, we graph $f(x) = x^3 - x$. Then we note that whenever the function changes sign, its graph passes through an x-intercept. Thus to solve $x^3 - x > 0$, we first solve the related equation $x^3 - x = 0$ to find all zeros of the function:

$$\begin{aligned} x^3 - x &= 0 \\ x(x^2 - 1) &= 0 \\ x(x + 1)(x - 1) &= 0. \end{aligned}$$

The zeros are -1, 0, and 1. Thus the x-intercepts of the graph are $(-1, 0)$, $(0, 0)$, and $(1, 0)$, as shown in the figure at left. The zeros divide the x-axis into four intervals:

$$(-\infty, -1), \qquad (-1, 0), \qquad (0, 1), \quad \text{and} \quad (1, \infty).$$

$-1 \quad 0 \quad 1 \quad x$

For all x-values within a given interval, the sign of $x^3 - x$ must be either *positive* or *negative*. To determine which, we choose a test value for x from each interval and find $f(x)$. We can use the TABLE feature set in ASK mode to determine the sign of $f(x)$ in each interval (see the table at left). We can also determine the sign of $f(x)$ in each interval by simply looking at the graph of the function.

INTERVAL	TEST VALUE	SIGN OF $f(x)$
$(-\infty, -1)$	$f(-2) = -6$	Negative
$(-1, 0)$	$f(-0.5) = 0.375$	Positive
$(0, 1)$	$f(0.5) = -0.375$	Negative
$(1, \infty)$	$f(2) = 6$	Positive

$- \quad + \quad - \quad +$

$-1 \quad 0 \quad 1 \quad x$

Since we are solving $x^3 - x > 0$, the solution set consists of only two of the four intervals, those in which the sign of $f(x)$ is *positive*. We see that the solution set is $(-1, 0) \cup (1, \infty)$, or $\{x \mid -1 < x < 0 \text{ or } x > 1\}$.

Shown in the box at left is a method for solving polynomial inequalities.

Rational Inequalities

Some inequalities involve rational expressions and functions. These are called **rational inequalities.** To solve rational inequalities, we need to make some adjustments to the preceding method.

EXAMPLE 3 Solve: $\dfrac{x-3}{x+4} \geq \dfrac{x+2}{x-5}$.

Solution We first subtract $(x+2)/(x-5)$ in order to find an equivalent inequality with 0 on one side:

$$\frac{x-3}{x+4} - \frac{x+2}{x-5} \geq 0.$$

Algebraic Solution

We look for all values of x for which the related function

$$f(x) = \frac{x-3}{x+4} - \frac{x+2}{x-5}$$

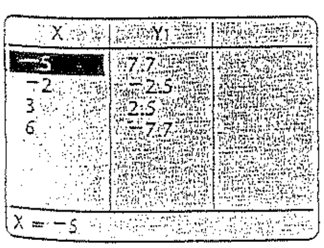
is not defined or is 0. These are called **critical values.**

A look at the denominators shows that $f(x)$ is not defined for $x = -4$ and $x = 5$. Next, we solve $f(x) = 0$:

$$\begin{aligned}
\frac{x-3}{x+4} - \frac{x+2}{x-5} &= 0 \\
(x+4)(x-5)\left(\frac{x-3}{x+4} - \frac{x+2}{x-5}\right) &= (x+4)(x-5)\cdot 0 \\
(x-5)(x-3) - (x+4)(x+2) &= 0 \\
(x^2 - 8x + 15) - (x^2 + 6x + 8) &= 0 \\
-14x + 7 &= 0 \\
x &= \tfrac{1}{2}.
\end{aligned}$$

The critical values are -4, $\frac{1}{2}$, and 5. These values divide the x-axis into four intervals:

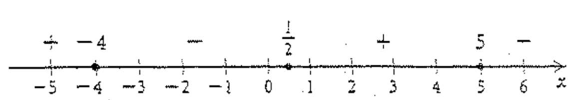
$$(-\infty, -4), \quad \left(-4, \tfrac{1}{2}\right), \quad \left(\tfrac{1}{2}, 5\right), \quad \text{and} \quad (5, \infty).$$



X	Y1
-5	7.7
-2	-2.5
3	2.5
6	-7.7

X = -5

We then use a test value to determine the sign of $f(x)$ in each interval.



Function values are positive in the intervals $(-\infty, -4)$ and $\left(\frac{1}{2}, 5\right)$. Since $f\left(\frac{1}{2}\right) = 0$ and the inequality symbol is $\geq$, we know that $\frac{1}{2}$ must be in the solution set. Note that since neither -4 nor 5 is in the domain of f, they cannot be part of the solution set.

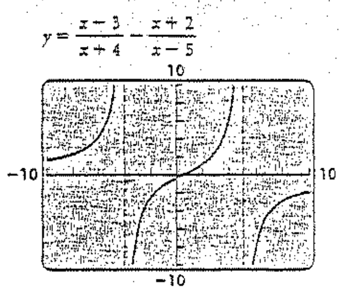
The solution set is $(-\infty, -4) \cup \left[\frac{1}{2}, 5\right)$.

Graphical Solution

We graph

$$y = \frac{x-3}{x+4} - \frac{x+2}{x-5}$$

in the standard window, which shows its curvature.



By using the ZERO feature, we find that 0.5 is a zero.

We then look for values where the function is not defined. By examining the denominators $x + 4$ and $x - 5$, we see that $f(x)$ is not defined for $x = -4$ and $x = 5$.

The **critical values,** where y is either not defined or 0, are -4, 0.5, and 5.

The graph shows where y is positive and where it is negative. Note that -4 and 5 cannot be in the solution set since y is not defined for these values. We do include 0.5, however, since the inequality symbol is $\geq$ and $f(0.5) = 0$. The solution set is

$$(-\infty, -4) \cup [0.5, 5).$$

The following is a method for solving rational inequalities.

To solve a rational inequality:

1. Find an equivalent inequality with 0 on one side.
2. Change the inequality symbol to an equals sign and solve the related equation.
3. Find values of the variable for which the related rational function is not defined.
4. The numbers found in steps (2) and (3) are called critical values. Use the critical values to divide the x-axis into intervals. Then test an x-value from each interval to determine the function's sign in that interval.
5. Select the intervals for which the inequality is satisfied and write interval notation or set-builder notation for the solution set. If the inequality symbol is $\leq$ or $\geq$, then the solutions to step (2) should be included in the solution set. The x-values found in step (3) are never included in the solution set.

It works well to use a combination of algebraic and graphical methods to solve polynomial and rational inequalities. The algebraic methods give exact numbers for the critical values, and the graphical methods allow us to see easily what intervals satisfy the inequality.

4

II. Fractional Exponents/Logarithms/Log Properties

Fractional Exponents

$$x^{\frac{a}{b}} = \sqrt[b]{x^a} = \left(\sqrt[b]{x}\right)^a$$

Examples:

1. $32^{\frac{3}{5}} = \left(\sqrt[5]{32}\right)^3 = 2^3 = 8$
2. $125^{-\frac{2}{3}} = \dfrac{1}{125^{\frac{2}{3}}} = \left(\dfrac{1}{\sqrt[3]{125}}\right)^2 = \left(\dfrac{1}{5}\right)^2 = \dfrac{1}{25}$

Logarithms

EXAMPLE 1 Computing Logarithms

Compute each of the following quantities.

a. $\log_2 8$ b. $\log_{1/3} 9$

SOLUTION

a. We need to find the power to which 2 must be raised in order to get 8. Since $2^3 = 8$, $\log_2 8 = 3$.

b. We are interested in finding the power to which $\frac{1}{3}$ must be raised in order to get 9. Since $(\frac{1}{3})^{-2} = 9$, $\log_{1/3} 9 = -2$.

From the definition of logarithm, $y = \log_a x$ if and only if $a^y = x$. In other words, for every logarithmic equation, there is a corresponding exponential equation, and vice versa. This equivalence will prove to be a useful tool both for solving equations involving logarithms and for deriving logarithmic identities.

Equivalence of Exponential and Logarithmic Equations

> The logarithmic equation $y = \log_a x$ and the exponential equation $a^y = x$ are equivalent.

The following examples illustrate the equivalence of exponential and logarithmic equations.

EXAMPLE 3 Writing Logarithmic Equations as Exponential Equations

Write an equivalent exponential equation for each of the following logarithmic equations.

a. $\log_{10} 1000 = 3$ b. $\log_x 100 = 2.1$ c. $\log_3(x^2) = 12$

SOLUTION

a. $10^3 = 1000$
b. $x^{2.1} = 100$
c. $3^{12} = x^2$

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Properties of Common and Natural Logarithms

- $10^{\log x} = x$ and $e^{\ln x} = x$
- $\log(10^x) = x$ and $\ln(e^x) = x$
- $\log 1 = \ln 1 = 0$
- The domain of both $f(x) = \log x$ and $g(x) = \ln x$ is the set of positive real numbers.

EXAMPLE 6 **Computing Common and Natural Logarithms**

Compute each of the following quantities, using a calculator as necessary.

a. $\log 1000$ b. $\ln \frac{1}{e}$

SOLUTION

a. Because the base isn't given, it is understood to be 10. Thus, we must find the power to which 10 is raised in order to get 1000. Since $10^3 = 1000$, $\log 1000 = 3$.

b. Here the base is e. We must find the power to which e is raised in order to get $1/e$. Since $e^{-1} = 1/e$, $\ln(1/e) = -1$.

EXAMPLE 4 **Simplifying with Logarithmic Identities**

Express $2 \ln x - \ln y + 6 \ln z$ as a single logarithm.

SOLUTION

$$\begin{aligned} 2\ln x - \ln y + 6\ln z &= \ln(x^2) - \ln y + \ln z^6 && \text{Identity 3} \\ &= \ln\left(\frac{x^2}{y}\right) + \ln(z^6) && \text{Identity 2} \\ &= \ln\left(\frac{x^2 z^6}{y}\right) && \text{Identity 1} \end{aligned}$$

LOGARITHMIC EQUATIONS

Equations involving one or more logarithmic expressions are called **logarithmic equations**. Many logarithmic equations can be solved by converting to exponential equations. For example, the equation

$$\ln x = 2$$

is equivalent to

$$x = e^2$$

This solution could also be found by exponentiating both sides of the original equation, as follows:

$$\begin{aligned} \ln x &= 2 \\ e^{\ln x} &= e^2 && \text{Exponentiating both sides} \\ x &= e^2 && \text{Using the property } a^{\log_a x} = x \end{aligned}$$

PIECEWISE-DEFINED FUNCTIONS

Occasionally, it is desirable to use two or more formulas to define a function. Consider, for example, the function f defined by

$$f(x) = \begin{cases} -x, & x \leq -1 \\ 2x + 3, & x > -1 \end{cases}$$

This *piecewise* definition indicates that for an input value x less than or equal to -1, the output value is $-x$. For an input value x greater than -1, the output value is $2x + 3$. Thus, $f(-4) = -(-4) = 4$, whereas $f(1) = 2(1) + 3 = 5$. Some additional input–output pairs are given in Table 8. Notice that for each $x \leq -1$, $f(x)$ is computed

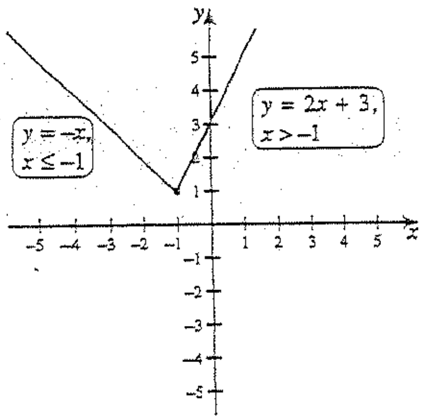

FIGURE 88

TABLE 8

x	Interval	Formula	$f(x)$
-3	$x \leq -1$	$-x$	$-(-3) = 3$
-2	$x \leq -1$	$-x$	$-(-2) = 2$
-1	$x \leq -1$	$-x$	$-(-1) = 1$
0	$x > -1$	$2x + 3$	$2(0) + 3 = 3$
1	$x > -1$	$2x + 3$	$2(1) + 3 = 5$
2	$x > -1$	$2x + 3$	$2(2) + 3 = 7$

using $-x$, whereas for $x > 1$, $f(x)$ is computed using $2x + 3$. The graph of f is obtained by plotting the line $y = -x$ for $x \leq -1$, and the line $y = 2x + 3$ for $x > -1$, as shown in Figure 88.

EXAMPLE 4 Evaluating and Graphing a Piecewise-Defined Function

Evaluate the function

$$f(x) = \begin{cases} x + 1, & x < 2 \\ x^2 - 4, & x \geq 2 \end{cases}$$

at the x-values given in the following table, and then sketch its graph.

x	$f(x)$
-1	
0	
1	
2	
3	

SOLUTION In order to compute a function value for a given x, we simply note whether $x < 2$ or $x \geq 2$ and use the corresponding "piece" of the function definition. For example, to compute $f(-1)$, we note that $-1 < 2$, and so we use the $x + 1$ piece of the function to obtain $f(-1) = (-1) + 1 = 0$. Similarly, since $3 \geq 2$, we compute $f(3)$ using the $x^2 - 4$ piece, obtaining $f(3) = (3)^2 - 4 = 5$. The remaining output values are computed in Table 9. The graph of f is obtained by plotting the line $y = x + 1$ for $x < 2$ and the parabola $y = x^2 - 4$ for $x \geq 2$, as shown in Figure 89. Notice that an open circle is used at the point $(2, 3)$ to indicate that the point is not included on the graph, and a closed circle is used at $(2, 0)$ to indicate that this point is included.

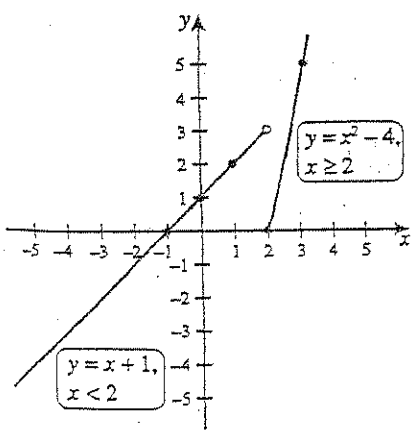

FIGURE 89

TABLE 9

x	Interval	Formula	$f(x)$
-1	$x < 2$	$x + 1$	$(-1) + 1 = 0$
0	$x < 2$	$x + 1$	$(0) + 1 = 1$
1	$x < 2$	$x + 1$	$(1) + 1 = 2$
2	$x \geq 2$	$x^2 - 4$	$(2)^2 - 4 = 0$
3	$x \geq 2$	$x^2 - 4$	$(3)^2 - 4 = 5$

WARNING! A piecewise-defined function is *one* function with several pieces, not several different functions. Thus, for a given input value x, the corresponding output value $f(x)$ is computed using only one of the pieces.

DOMAIN

EXAMPLE 9 **Finding Domain**

Find the domain of $g(t) = \dfrac{t}{t^2 - 1}$.

SOLUTION All input values of t are valid except those that lead to a zero in the denominator. To determine where this occurs, we set $t^2 - 1 = 0$ and solve for t.

$$t^2 - 1 = 0$$
$$(t + 1)(t - 1) = 0$$
$$t = -1, t = 1$$

So the domain of g is the set of all real numbers t except -1 and 1. This set can be written in set-builder notation as $\{t \mid t \neq -1, t \neq 1\}$. In interval notation, we write $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

EXAMPLE 10 **Finding Domain**

Find the domain of $h(x) = \sqrt{2x - 3}$.

SOLUTION Since we are only considering values of x for which $h(x)$ is real, the valid input values of x are those for which $2x - 3 \geq 0$. Solving this linear inequality for x yields $x \geq \frac{3}{2}$. Thus, the domain of h is $[\frac{3}{2}, \infty)$.

RATIONAL FUNCTIONS

A **rational function** is a function of the form

$$f(x) = \frac{p(x)}{q(x)}$$

where p and q are polynomials. Examples include

$$g(x) = \frac{2x + 3}{x^2 - 4} \quad \text{and} \quad h(x) = \frac{2}{x - 5}$$

as well as all polynomial functions, such as $f(x) = x^2 + 3x$, where the denominator is assumed to be 1. In this section, we only consider rational functions *in lowest terms*—that is, rational functions $f(x) = p(x)/q(x)$ for which $p(x)$ and $q(x)$ have no common factors. See Exercise 62 for a discussion of rational functions that are not in lowest terms.

DOMAIN AND ZEROS

Since a rational function f is constructed from polynomials p and q, it is not surprising that key features of f—its domain, x-intercepts, end behavior, and so forth—prove to be closely tied to the behaviors of these polynomials. In particular, since $f(x) = p(x)/q(x)$, $f(x)$ is defined only where both $p(x)$ and $q(x)$ are defined and where the denominator $q(x)$ is nonzero. Thus, because polynomials are defined for all real numbers, the domain of a rational function is found by excluding the zeros of the polynomial in the denominator.

Domain of a Rational Function

The domain of a rational function $f(x) = p(x)/q(x)$ consists of all real numbers x such that $q(x) \neq 0$.

We've seen that zeros of the denominator q determine the domain of f. On the other hand, since a fraction is only zero when its numerator is zero, the zeros of f correspond to the zeros of its numerator.

Zeros of a Rational Function

If $f(x) = p(x)/q(x)$ is a rational function in lowest terms, then $f(x) = 0$ if and only if $p(x) = 0$.

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HORIZONTAL ASYMPTOTES

Determine the horizontal asymptotes using the three cases below.

Case I. Degree of the numerator is less than the degree of the denominator. The asymptote is y = 0.

Example: $y=\dfrac{1}{x-1}$ (As x becomes very large or very negative the value of this function will approach 0). Thus there is a horizontal asymptote at $y=0$.

Case II. Degree of the numerator is the same as the degree of the denominator. The asymptote is the ratio of the lead coefficients.

Exmaple: $y=\dfrac{2x^2+x-1}{3x^2+4}$ (As x becomes very large or very negative the value of this function will approach 2/3). Thus there is a horizontal asymptote at $y=\dfrac{2}{3}$.

Case III. Degree of the numerator is greater than the degree of the denominator. There is no horizontal asymptote. The function increases without bound. (If the degree of the numerator is exactly 1 more than the degree of the denominator, then there exists a slant asymptote, which is determined by long division.)

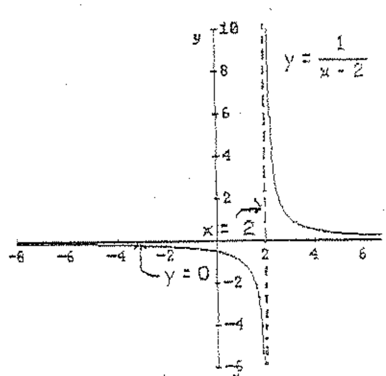
Example: $y=\dfrac{2x^2+x-1}{3x-3}$ (As x becomes very large the value of the function will continue to increase and as x becomes very negative the value of the function will also become more negative).

VERTICAL ASYMPTOTES

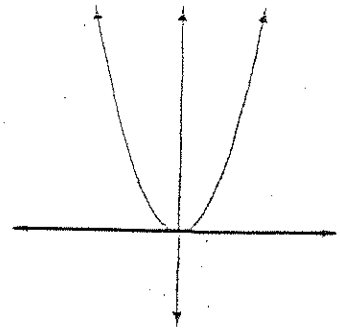
Determine the vertical asymptotes for the function. Set the denominator equal to zero to find the x-value for which the function is undefined. That will be the vertical asymptote given the numerator does not equal 0 also (Remember this is called removable discontinuity).
Write a vertical asymptotes as a line in the form x =

Example: Find the vertical asymptote of $y=\dfrac{1}{x-2}$

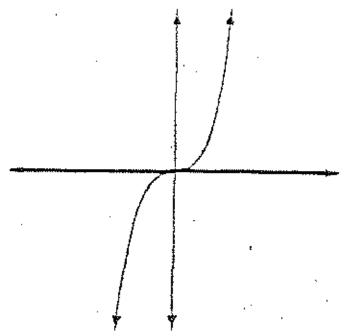
Since when $x=2$ the function is in the form 1/0 then the vertical line $x=2$ is a vertical asymptote of the function.



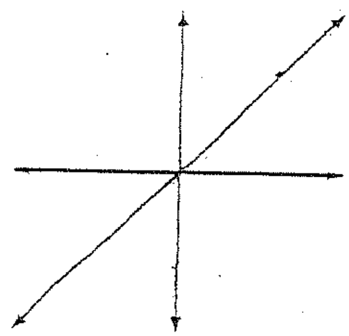
Important Functions to Memorize



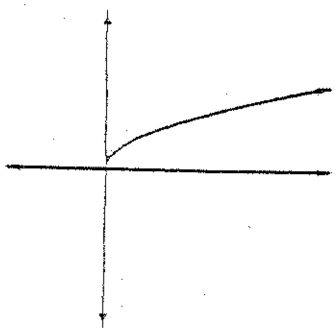
$y = x^2$
Domain: $(-\infty,\infty)$
Range: $[0,\infty)$



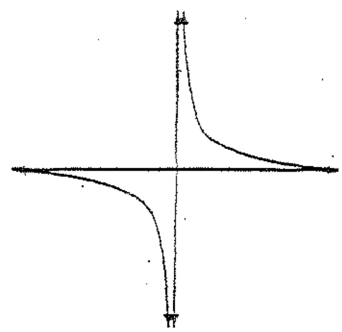
$y = x^3$
Domain: $(-\infty,\infty)$
Range: $(-\infty,\infty)$



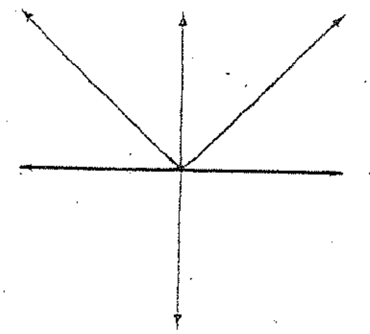
$y = x$
Domain: $(-\infty,\infty)$
Range: $(-\infty,\infty)$



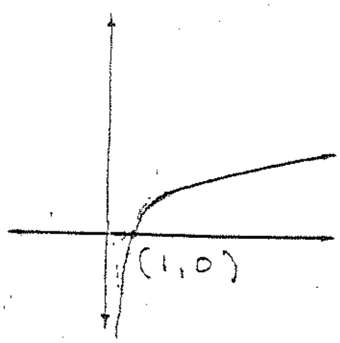
$y = \sqrt{x}$
Domain: $[0,\infty)$
Range: $[0,\infty)$



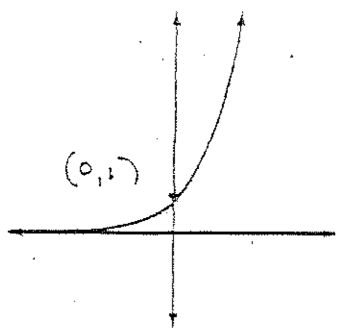
$y = \frac{1}{x}$
Domain: $(-\infty,0)\cup(0,\infty)$
Range: $(-\infty,0)\cup(0,\infty)$



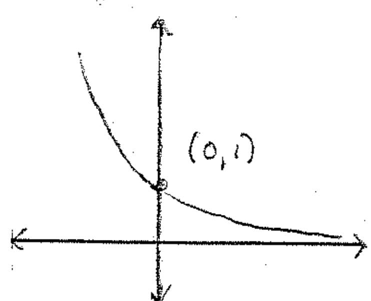
$y = |x|$
Domain: $(-\infty,\infty)$
Range: $[0,\infty)$



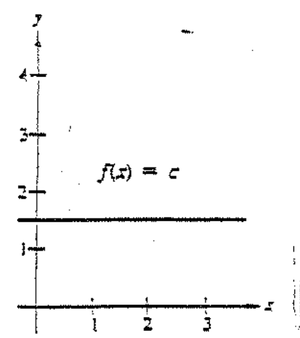
$y = \ln x$
Domain: $(0,\infty)$
Range: $(-\infty,\infty)$
Contains point (1,0)



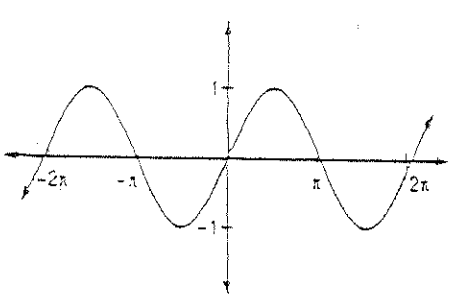
$y = e^x$
Domain: $(-\infty,\infty)$
Range: $(0,\infty)$
Contains point (0,1)



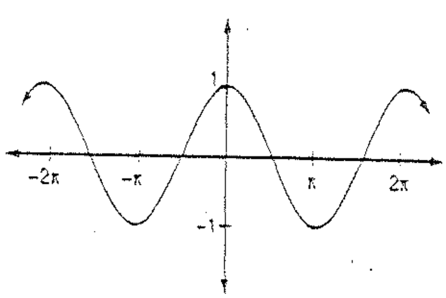
$y = e^{-x}$
Domain: $(-\infty,\infty)$
Range: $(0,\infty)$



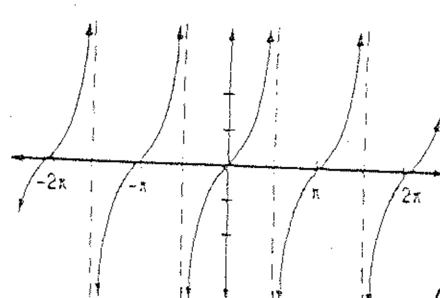
Constant Function



$y = \sin x$



$y = \cos x$



$y = \tan x$