Curvature intuition https://www.youtube.com/watch?v=ugtUGhBSeE0

Curvature formula, part 1 https://www.youtube.com/watch?v=gspjhwSNMWs

Parametric Graphing Calculator:
https://christopherchudzicki.github.io/MathBox-Demos/parametric_curves_3D.html

Curvature a measure of how much a curve bends

Remember unit tangent vector

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

Curvature would then be the rate at which the direction along the curve is changing i.e. magnitude of the unit tangent vector, BUT it would depend on how fast you "walk" along the path. The faster you walk the quicker the vector will change.

We will want to walk at unit speed → arc length parametrization!

$\mathbf{r}(s)$ is an arc length parametrization if $|\mathbf{r}(s)| = 1$ for all s

> **Curvature** Let $\mathbf{r}(s)$ be an arc length parametrization and $\mathbf{T}$ the unit tangent vector. The curvature of the underlying curve at $\mathbf{r}(s)$ is the quantity denoted by "kappa"
>
> $$\kappa(t) = \left|\frac{d\mathbf{T}}{ds}\right|$$

1. Compute the curvature of a circle of radius R (assume centered at the origin, so it has parametrization $\mathbf{r}(\theta) = \langle R\cos\theta, R\sin\theta\rangle$

$r'(\theta) = \langle -R\sin\theta, R\cos\theta\rangle$

$$s(\theta) = \int_0^\theta \sqrt{(-R\sin\theta)^2 + (R\sin\theta)^2}\, d\theta$$

$$= \int_0^\theta \sqrt{R^2}\, d\theta$$

$$= \int_0^\theta R\, d\theta = R\theta \qquad s = R\theta \quad \theta = s/R$$

$r(s) = \langle R\cos(s/R), R\sin(s/R)\rangle$

$T(s) = r'(s) = \langle -\sin s/R, \cos s/R\rangle$

$\frac{dT}{ds} = T'(s) = \langle -\frac{1}{R}\cos\frac{s}{R}, -\frac{1}{R}\sin\frac{s}{R}\rangle$

$\kappa(s) = \left|\frac{dT}{ds}\right| = \sqrt{(-\frac{1}{R}\cos\frac{s}{R})^2 + (-\frac{1}{R}\sin\frac{s}{R})^2}$

In practice, it is often difficult if not impossible to find arc length parametrization explicitly. We can derive other formulas for curvature that do not require arc length parametrization. Check your textbook for the proofs for these formulas!

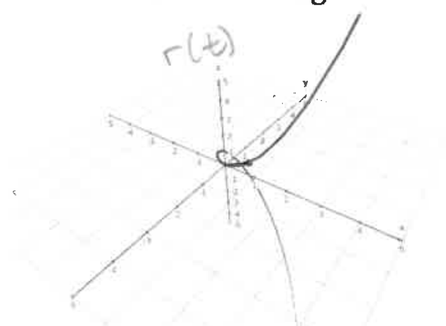
$= \sqrt{\frac{1}{R^2}\cos^2\frac{s}{R} + \frac{1}{R^2}\sin^2\frac{s}{R}}$

$= \sqrt{1/R^2} = \boxed{1/R}$

Curvature If $\mathbf{r}(t)$ is a regular parametrization, then the curvature of the underlying curve at $\mathbf{r}(t)$ is

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

2. Calculate the curvature $\kappa(t)$ of the twisted cubic $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ graphed below. Then graph $\kappa(t)$ on a calculator and determine whether the curvature is largest.

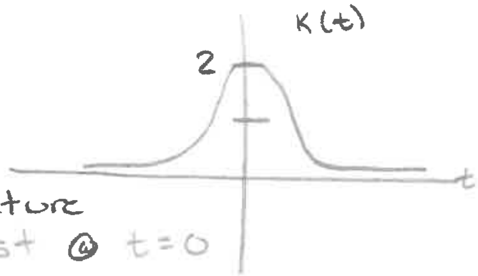


$r'(t) = \langle 1, 2t, 3t^2 \rangle$

$r''(t) = \langle 0, 2, 6t \rangle$

$$r'(t) \times r''(t) = \begin{vmatrix} i & j & k \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix}$$

$$= 6t^2 i - 6t j + 2k$$



curvature largest @ $t=0$

$$K(t) = \frac{\sqrt{36t^4 + 36t^2 + 4}}{\left(\sqrt{1 + 4t^2 + 9t^4}\right)^3}$$

Curvature of a Plane Curve (plane curve → is a curve that lies in a single plane)

Assume $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ is a regular parametrization of a plane curve. At the point $(x(t),\ y(t))$, the curvature is given by

$$\kappa(t) = \frac{|x'(t)y''(t) - y'(t)x''(t)|}{((x'(t))^2 + (y'(t))^2)^{3/2}}$$

absolute value not magnitude

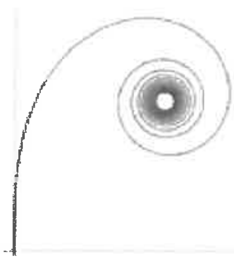
3. Highway engineers design roads to achieve continuous and simple transitions between road segments with different curvatures, such as between the straight highways and the circular parts of the entrance and exit ramps. Curve segments commonly used in such transitions are taken from the **Cornu spiral** that is defined parametrically by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, $t \geq 0$, where

$$x(t) = \int_0^t \sin u^2 du \qquad y(t) = \int_0^t \cos u^2 du$$

Show that $\kappa(t) = 2t$, and therefore curvature changes linearly as a function of t along the Cornu spiral.

$x'(t) = \sin t^2$ $\qquad$ $y'(t) = \cos t^2$

$x''(t) = 2t \cos t^2$ $\qquad$ $y''(t) = -2t \sin t^2$



$x'(t)y''(t) - y'(t)x''(t) = -2t\sin^2 t^2 - 2t\cos^2 t^2$

$= -2t$

$(x'(t))^2 + (y'(t))^2 = \sin^2 t^2 + \cos^2 t^2 = 1$

$$\kappa(t) = \frac{|-2t|}{1^{3/2}} = 2t$$

Curvature of the Graph of f

The curvature at the point $(x, f(x))$ on the graph of $y = f(x)$ is equal to

$$\kappa(t) = \frac{|f''(x)|}{(1 + (f'(x))^2)^{3/2}}$$

absolute value not magnitude

4. Compute the curvature of $f(x) = x^3 - 3x^2 + 4$ at $x = 0, 1, 2,$ and 3.

$f'(x) = 3x^2 - 6x$

$f''(x) = 6x - 6$

$$\kappa(x) = \frac{|6x - 6|}{(1 + (3x^2 - 6x)^2)^{3/2}}$$

$\kappa(0) = \frac{6}{1} = 6$

$\kappa(2) = \frac{6}{(1 + (12 - 12)^2)^{3/2}} = 6$

$\kappa(1) = 0$

$\kappa(3) = \frac{12}{(1 + (27 - 18)^2)^{3/2}} \approx 0.016$