Normal Vector
In 14.2 we proved $\mathbf{r}(t)$ is orthogonal to $\mathbf{r}'(t)$, which can be extended to $\mathbf{T}'(t)$ and $\mathbf{T}(t)$. The unit vector in the direction of $\mathbf{T}'(t)$ is called the **normal vector** and denoted $\mathbf{N}(t)$.

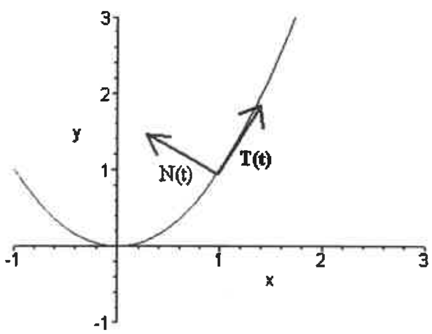
$$T(t) = \frac{r'(t)}{\|r'(t)\|}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$$

or

$$\mathbf{T}'(t) = v(t)\kappa(t)\mathbf{N}(t)$$

For a plane curve, $\mathbf{N}(t)$ points in the direction of bending



1. Consider the helix

$$\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), t\rangle$$

Show that for all t, the normal vector is parallel to the xy —plane and points toward the z-axis.

$$r'(t) = \langle -2\sin t, 2\cos t, 1\rangle$$

$$\|r'(t)\| = \sqrt{(-2\sin t)^2 + (2\cos t)^2 + 1^2}$$

$$= \sqrt{4(\sin^2 t + \cos^2 t) + 1}$$

$$= \sqrt{5}$$

$$T(t) = \frac{1}{\sqrt{5}}\langle -2\sin t, 2\cos t, 1\rangle$$

$$T'(t) = \frac{1}{\sqrt{5}}\langle -2\cos t, -2\sin t, 0\rangle$$

$$\|T'(t)\| = \frac{1}{\sqrt{5}}\sqrt{(-2\cos t)^2 + (2\sin t)^2}$$

$$= \frac{2}{\sqrt{5}}$$

continued →

Binormal Vector

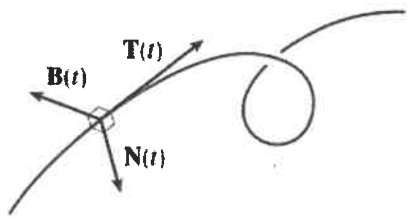
At a point P on a curve, the vectors **T** and **N** determine a plane. The normal vector to this plane, **B (Binormal Vector)** is defined by

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}$$

B is orthogonal to both **T** and **N**

B is a unit vector since $\|\mathbf{B}\| = \|\mathbf{T}\|\,\|\mathbf{N}\|\sin(\pi/2) = (1)(1)(1) = 1$

Frenet Frame the set of mutually perpendicular vectors **B, T,** and **N.**



2. Determine the binormal vector for the helix

$$\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), t\rangle$$

* T(t) and N(t) from example 1 *

$T(t) = \frac{1}{\sqrt{5}}\langle -2\sin t,\ 2\cos t,\ 1\rangle$ $\qquad N(t) = \langle -\cos t, -\sin t, 0\rangle$

$$B(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\frac{2}{\sqrt{5}}\sin t & \frac{2}{\sqrt{5}}\cos t & 1/\sqrt{5} \\ -\cos t & -\sin t & 0 \end{vmatrix} = \frac{1}{\sqrt{5}}\sin t\,\vec{i} - \frac{1}{\sqrt{5}}\cos t\,\vec{j} + \frac{2}{\sqrt{5}}\vec{k}$$

3. Find the normal and binormal vectors for

$$\vec{r}(t) = \langle t, 3\sin t, 3\cos t\rangle.$$

$r'(t) = \langle 1, 3\cos t, -3\sin t\rangle$

$\|r'(t)\| = \sqrt{1 + 9\cos^2 t + 9\sin^2 t}$

$= \sqrt{10}$

$T(t) = \frac{1}{\sqrt{10}}\langle 1, 3\cos t, -3\sin t\rangle$

$T'(t) = \langle 0, -\frac{3}{\sqrt{10}}\sin t, -\frac{3}{\sqrt{10}}\cos t\rangle$

$\|T'(t)\| = \sqrt{9/10\sin^2 t + 9/10\cos^2 t}$

$= 3/\sqrt{10}$

continued →

1)

$$N(t) = \frac{\frac{1}{\sqrt{5}}\langle -2\cos t, -2\sin t, 0\rangle}{\frac{2}{\sqrt{5}}}$$

$$= \frac{1}{2}\langle -2\cos t, -2\sin t, 0\rangle$$

$$= \langle -\cos t, -\sin t, 0\rangle$$

since z-component is 0, N(t) is parallel to xy-plane

since x & y components of N(t) are both <u>negative</u> scalar multiples of r(t), N(t) points towards the z-axis

3)

$$N(t) = \frac{\langle 0, -3/\sqrt{10}\sin t, -3/\sqrt{10}\cos t \rangle}{3/\sqrt{10}}$$

$$= \boxed{\langle 0, -\sin t, -\cos t \rangle}$$

$$B(t) = \begin{vmatrix} i & j & k \\ 1/\sqrt{10} & 3/\sqrt{10}\cos t & -3/\sqrt{10}\sin t \\ 0 & -\sin t & -\cos t \end{vmatrix}$$

$$= -\frac{3}{\sqrt{10}}\cos^2 t\,\vec{i} + \frac{1}{\sqrt{10}}\cos t\,\vec{j} - \frac{1}{\sqrt{10}}\sin t\,\vec{k}$$

$$-\frac{3}{\sqrt{10}}\sin^2 t\,\vec{i}$$

$$= \boxed{-\frac{3}{\sqrt{10}}\vec{i} + \frac{1}{\sqrt{10}}\cos t\,\vec{j} - \frac{1}{\sqrt{10}}\sin t\,\vec{k}}$$