Fubini's Theorem for Triple Integrals: The triple integral of a continuous function $f(x, y, z)$ over a box $B = [a, b] \times [c, d] \times [p, q]$ is equal to the iterated integral:

$$\iiint_B f(x,y,z)dV = \int_{x=a}^{b} \int_{y=c}^{d} \int_{z=p}^{q} f(x,y,z)dzdydx$$

Furthermore, the iterated integral may be evaluated in any order.

1. Evaluate $\iiint_B x^2 e^{y+3z} dV$, where $B = \overset{x}{[1,4]} \times \overset{y}{[0,3]} \times \overset{z}{[2,6]}$

order integration any way you want that makes sense for function

$$\int_1^4 \int_0^3 \int_2^6 x^2 e^{y+3z} \, dz\, dy\, dx$$

$$\int_1^4 \int_0^3 \frac{1}{3}x^2 e^{y+3z} \Big|_2^6 \, dy\, dx$$

$$\int_1^4 \int_0^3 \frac{1}{3}x^2 e^{y+18} - \frac{1}{3}x^2 e^{y+6} \, dy\, dx$$

$$\int_1^4 \frac{1}{3}x^2 e^{y+18} - \frac{1}{3}x^2 e^{y+6} \Big|_0^3 \, dx$$

$$\int_1^4 \frac{1}{3}x^2 e^{21} - \frac{1}{3}x^2 e^{9} - \frac{1}{3}x^2 e^{18} + \frac{1}{3}x^2 e^{6} \, dx$$

$$\frac{1}{9}x^3 (e^{21} - e^9 - e^{18} + e^6) \Big|_1^4$$

$$\frac{64}{9}(e^{21} - e^9 - e^{18} + e^6) - \frac{1}{9}(e^{21} - e^9 - e^{18} + e^6)$$

$$\boxed{7(e^{21} - e^9 - e^{18} + e^6)}$$

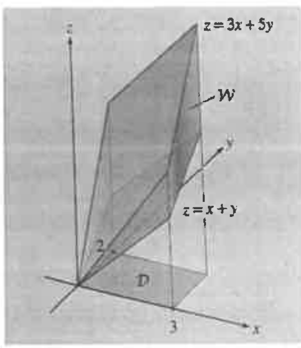
The triple integral of a continuous function f over the region

$$W:\ (x, y) \in D,\ \ z_1(x, y) \le z \le z_2(x, y)$$

Is equal to the iterated integral

$$\iiint_W f(x, y, z)\,dV = \iint_D \left(\int_{z=z_1(x,y)}^{z=z_2(x,y)} f(x, y, z)\,dz \right) dA$$

2. Evaluate $\iiint_W z\,dV$, where W is the region between the planes $z = x + y$ and $z = 3x + 5y$ lying over the rectangle $D = [0, 3] \times [0, 2]$



$$\int_0^3 \int_0^2 \int_{x+y}^{3x+5y} z \, dz\, dy\, dx$$

$$\int_0^3 \int_0^2 \frac{1}{2} z^2 \Big|_{x+y}^{3x+5y} dy\, dx$$

$$\int_0^3 \int_0^2 \frac{1}{2}(3x+5y)^2 - \frac{1}{2}(x+y)^2\, dy\, dx$$

$$\frac{1}{2}\int_0^3 \int_0^2 9x^2 + 30xy + 25y^2 - x^2 - 2xy - y^2\, dy\, dx$$

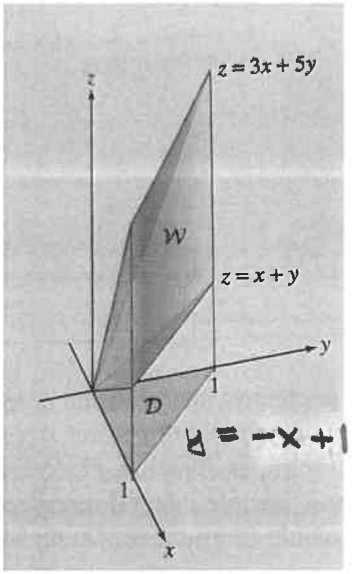
$$\int_0^3 \int_0^2 4x^2 + 14xy + 12y^2\, dy\, dx$$

$$\int_0^3 4x^2y + 7xy^2 + 4y^3 \Big|_0^2 dx$$

$$\int_0^3 8x^2 + 28x + 32\, dx$$

$$\frac{8}{3}x^3 + 14x^2 + 32x \Big|_0^3 = 72 + 126 + 96 = \boxed{294}$$

3. Evaluate $\iiint\limits_W z\,dV$, where W is the region between the planes $z = x + y$ and $z = 3x + 5y$ lying over the triangle shown in the figure below.



y & x no longer rectangular so need functions to bound

$0 < x < 1$

$0 < y < -x+1$

OR

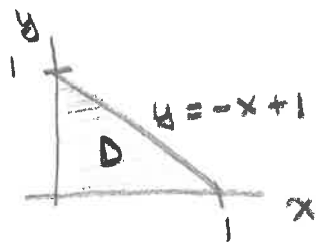
$0 < x < 1-y$

$0 < y < 1$

$$\int_0^1 \int_0^{-x+1} \int_{x+y}^{3x+5y} z \, dz \, dy \, dx$$

$$\int_0^1 \int_0^{-x+1} \frac{1}{2} z^2 \Big|_{x+y}^{3x+5y} dy \, dx$$

$$\int_0^1 \int_0^{-x+1} 4x^2 + 14xy + 12y^2 \, dy \, dx$$



$$\int_0^1 4x^2y + 7xy^2 + 4y^3 \Big|_0^{-x+1} dx$$

$$\int_0^1 4x^2(-x+1) + 7x(-x+1)^2 + 4(-x+1)^3 \, dx$$

$$\int_0^1 -4x^3 + 4x^2 + 7x^3 - 14x^2 + 7x + 4(-x+1)^3 \, dx$$

$$\int_0^1 3x^3 - 10x^2 + 7x + 4(-x+1)^3 \, dx$$

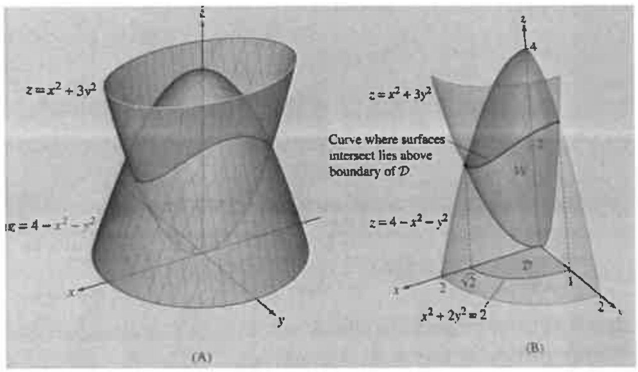
$$\frac{3}{4}x^4 - \frac{10}{3}x^3 + \frac{7}{2}x^2 - (-x+1)^4 \Big|_0^1$$

$$\frac{3}{4} - \frac{10}{3} + \frac{7}{2} - 0 - (-(1)^4)$$

$$\boxed{\frac{23}{12}}$$

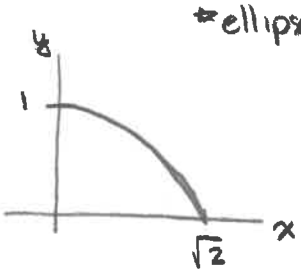
4. Integrate $f(x, y, z) = x$ over the region W bounded above by $z = 4 - x^2 - y^2$ and below by $z = x^2 + 3y^2$ in the octant $x \geq 0,\ y \geq 0,\ z \geq 0$

Step 1: Find the boundary of D



* find where the surfaces intersect
↳ think of this as a shadow down to xy plane

$$4 - x^2 - y^2 = x^2 + 3y^2$$
$$4 = 2x^2 + 4y^2$$
$$2 = x^2 + 2y^2$$
ellipse $1 = \frac{1}{2}x^2 + y^2$



can integrate in any order

$0 \leq y \leq 1$

$0 \leq x \leq \sqrt{2 - 2y^2}$

$$\int_0^1 \int_0^{\sqrt{2-2y^2}} \int_{x^2+3y^2}^{4-x^2-y^2} x \, dz \, dx \, dy$$

$$\int_0^1 \int_0^{\sqrt{2-2y^2}} x z \Big|_{x^2+3y^2}^{4-x^2-y^2} dx\, dy$$

$$\int_0^1 \int_0^{\sqrt{2-2y^2}} x(4 - x^2 - y^2) - x(x^2 + 3y^2)\, dx\, dy$$

$$\int_0^1 \int_0^{\sqrt{2-2y^2}} 4x - x^3 - xy^2 - x^3 - 3xy^2 \, dx\, dy$$

$$\int_0^1 \int_0^{\sqrt{2-2y^2}} -2x^3 - 4xy^2 + 4x \, dx\, dy$$

$$\int_0^1 -\frac{1}{2}x^4 - 2x^2y^2 + 2x^2 \Big|_0^{\sqrt{2-2y^2}} dy$$

$$\int_0^1 -\frac{1}{2}(2 - 2y^2)^2 - 2(2 - 2y^2)y^2 + 2(2 - 2y^2)\, dy$$

$$\int_0^1 -\frac{1}{2}(4-8y^2+4y^4)-4y^2+4y^4+4-4y^2\,dy$$

$$\int_0^1 -2+4y^2-2y^4-8y^2+4y^4+4\,dy$$

$$\int_0^1 2y^4-4y^2+2\,dy$$

$$\frac{2}{5}y^5-\frac{4}{3}y^3+2y\Big|_0^1$$

$$\frac{2}{5}-\frac{4}{3}+2$$

$$\boxed{\frac{16}{15}}$$