Remember:

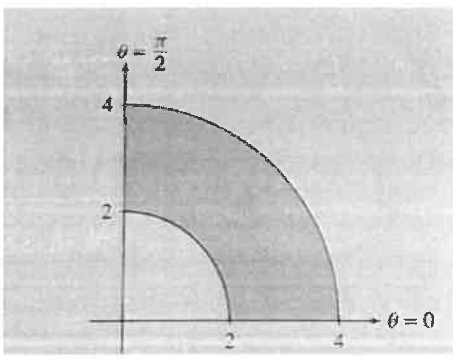
$x = r\cos\theta \qquad y = r\sin\theta$

Double Integral in Polar Coordinates For a continuous function f on the domain

$$D\colon \theta_1 \le \theta \le \theta_2,\ \ r_1(\theta) \le r \le r_2(\theta)$$

$$\iint_D f(x,y)\,dA = \int_{\theta_1}^{\theta_2}\int_{r=r_1(\theta)}^{r=r_2(\theta)} f(r\cos\theta,\ r\sin\theta)\,r\,dr\,d\theta$$

1. Compute $\iint_D (x+y)\,dA$, where D is the quarter annulus in the figure below:



Describe D in polar

$2 \le r \le 4 \qquad 0 \le \theta \le \pi/2$

$f(x,y) = x + y$ *rewrite in polar

$= r\cos\theta + r\sin\theta$

$= r(\cos\theta + \sin\theta)$

$$\int_0^{\pi/2}\int_2^4 r(\cos\theta + \sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_2^4 r^2(\cos\theta + \sin\theta)\,dr\,d\theta$$

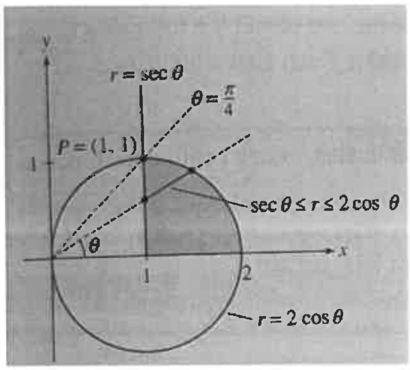
$$= \int_0^{\pi/2} \tfrac{1}{3}r^3(\cos\theta + \sin\theta)\Big|_2^4\,d\theta$$

$$= \tfrac{1}{3}\int_0^{\pi/2} 64(\cos\theta + \sin\theta) - 8(\cos\theta + \sin\theta)\,d\theta$$

$$= \frac{56}{3}\int_0^{\pi/2}(\cos\theta + \sin\theta)\,d\theta = \frac{56}{3}\left(\sin\theta - \cos\theta\right)\Big|_0^{\pi/2} = \frac{56}{3}\left[(1-0)-(0-1)\right]$$

$$= \boxed{112/3}$$

2. Calculate $\iint_D (x^2+y^2)^{-2}\,dA$ for the shaded domain D in the figure below:



$\sec\theta \le r \le 2\cos\theta \qquad 0 \le \theta \le \pi/4$

$$(x^2+y^2)^{-2} = ((r\cos\theta)^2 + (r\sin\theta)^2)^{-2}$$
$$= (r^2(\cos^2\theta + \sin^2\theta))^{-2}$$
$$= (r^2(1))^{-2}$$
$$= r^{-4}$$

$$\int_0^{\pi/4}\int_{\sec\theta}^{2\cos\theta} r^{-4}\, r\,dr\,d\theta$$
$$= \int_0^{\pi/4} -\frac{1}{2}r^{-2}\Big|_{\sec\theta}^{2\cos\theta} d\theta$$
$$= \int_0^{\pi/4} -\frac{1}{2}\left[(2\cos\theta)^{-2} - (\sec\theta)^{-2}\right] d\theta$$
$$= -\frac{1}{2}\int_0^{\pi/4}\left(\frac{1}{4\cos^2\theta} - \frac{1}{\sec^2\theta}\right) d\theta$$
$$= -\frac{1}{2}\int_0^{\pi/4}\left(\frac{1}{4}\sec^2\theta - \cos^2\theta\right) d\theta$$
$$= -\frac{1}{8}\tan\theta\Big|_0^{\pi/4} + \frac{1}{2}\int_0^{\pi/4}\frac{1}{2}\cos 2\theta + \frac{1}{2}d\theta$$
$$= -\frac{1}{8}\tan\theta + \frac{1}{8}\sin 2\theta + \frac{1}{4}\theta\Big|_0^{\pi/4}$$
$$= -\frac{1}{8} + \frac{1}{8} + \frac{\pi}{16} - (0)$$
$$= \boxed{\pi/16}$$

* Trig Identities *

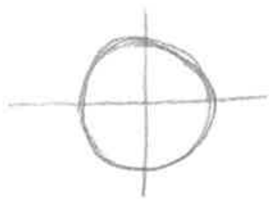
3. Evaluate $\iint_D 2xy\,dA$, D is the portion of the region between the circles of radius 2 and radius 5 centered at the origin that lies in the first quadrant.

$2 \le r \le 5 \qquad 0 \le \theta \le \pi/2$

$$2xy = 2r\cos\theta\, r\sin\theta$$
$$= 2r^2\cos\theta\sin\theta$$
$$= r^2\sin 2\theta$$

$$\int_0^{\pi/2}\int_2^5 r^2\sin 2\theta\, r\,dr\,d\theta$$
$$= \frac{1}{4}\int_0^{\pi/2} r^4\sin 2\theta\Big|_2^5 d\theta$$
$$= \frac{1}{4}\int_0^{\pi/2}(625\sin 2\theta - 16\sin 2\theta)\,d\theta$$
$$= \frac{609}{4}\int_0^{\pi/2}\sin 2\theta\,d\theta$$
$$= -\frac{609}{4}\left(\frac{1}{2}\cos 2\theta\right)\Big|_0^{\pi/2}$$
$$= -\frac{609}{8}(-1-1)$$
$$= \boxed{609/4}$$

4. Evaluate $\iint_D e^{x^2+y^2}\,dA$, D is the unit disk centered at the origin.



$0 \le r \le 1$

$0 \le \theta \le 2\pi$

$e^{x^2+y^2} = e^{r^2\cos^2\theta + r^2\sin^2\theta} = e^{r^2}$

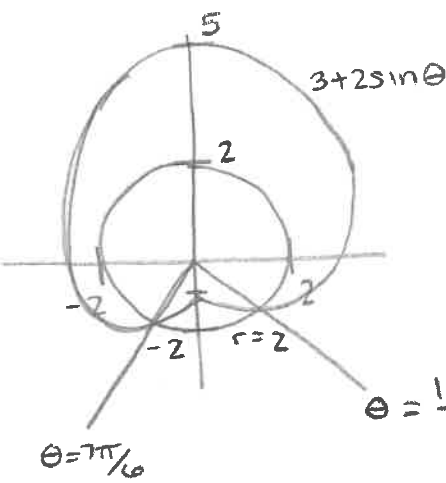
$\int_0^{2\pi}\int_0^1 e^{r^2} r\,dr\,d\theta$

$= \int_0^{2\pi} \frac{1}{2}e^{r^2}\Big|_0^1 d\theta$

$= \int_0^{2\pi}\left(\frac{1}{2}e - \frac{1}{2}\right)d\theta \rightarrow \frac{1}{2}e\theta - \frac{1}{2}\theta\Big|_0^{2\pi}$

$= \boxed{\pi e - \pi}$

5. Determine the area of the region that lies inside $r = 3 + 2\sin\theta$ and outside $r = 2$



$3 + 2\sin\theta = 2$

$\sin\theta = -\frac{1}{2}$

$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

$\theta = \frac{11\pi}{6} = -\frac{\pi}{6}$

$-\frac{\pi}{6} \le \theta \le \frac{7\pi}{6}$

$2 \le r \le 3 + 2\sin\theta$

$\iint_D dA = \int_{-\pi/6}^{7\pi/6}\int_2^{3+2\sin\theta} r\,dr\,d\theta$

$\int_{-\pi/6}^{7\pi/6} \frac{1}{2}r^2\Big|_2^{3+2\sin\theta} d\theta$

$\int_{-\pi/6}^{7\pi/6}\left[\frac{1}{2}(3+2\sin\theta)^2 - 2\right]d\theta$

$\int_{-\pi/6}^{7\pi/6}\left[9/2 + 2\sin^2\theta + 6\sin\theta - 2\right]d\theta$

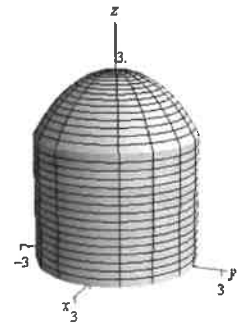
continued on paper

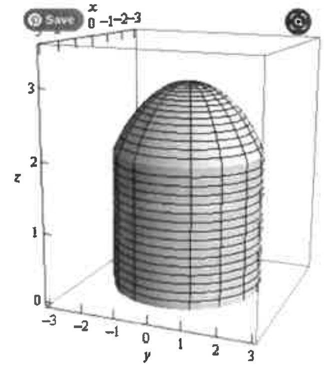
6. Determine the volume of the region that lies under the sphere $x^2 + y^2 + z^2 = 9$, above the plane $z = 0$ and inside the cylinder $x^2 + y^2 = 5$

*use geogebra to understand graphs

$z = \sqrt{9 - x^2 - y^2}$

* positive b/c above $z = 0$





$\sqrt{9 - (x^2 + y^2)} = \sqrt{9 - r^2}$

$x^2 + y^2 = 5$

$r^2 = 5$

$r = \pm\sqrt{5}$

$0 \leq r \leq \sqrt{5}$

$0 \leq \theta \leq 2\pi$

$$V = \int_0^{2\pi}\int_0^{\sqrt{5}} (9 - r^2)^{1/2}\, r\, dr\, d\theta$$

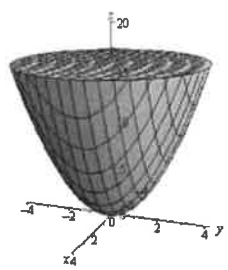
$$= \int_0^{2\pi} \frac{2}{3}(9 - r^2)^{3/2}\left(-\frac{1}{2}\right)\Big|_0^{\sqrt{5}}\, d\theta$$

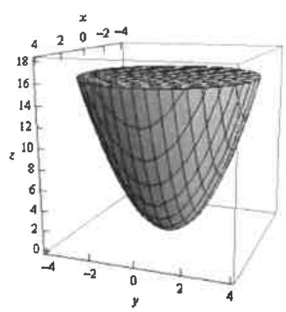
$$= -\frac{1}{3}\int_0^{2\pi} 4^{3/2} - 9^{3/2}\, d\theta$$

$8 - 27$

$$= \frac{19}{3}\theta\,\Big|_0^{2\pi} = \boxed{\frac{38\pi}{3}}$$

7. Find the volume of the region that lies inside $z = x^2 + y^2$ and below the plane $z = 16$





$16 - (x^2 + y^2) = 16 - r^2$

$$\int_0^{2\pi}\int_0^{4} (16 - r^2)\, r\, dr\, d\theta$$

$$\int_0^{2\pi} 8r^2 - \frac{1}{4}r^4\,\Big|_0^4\, d\theta$$

$16 = x^2 + y^2$

$16 = r^2$

$\pm 4 = r$

$0 \leq r \leq 4$

$0 \leq \theta \leq 2\pi$

$$\int_0^{2\pi} 8(16) - \frac{1}{4}(256)\, d\theta$$

$$64\theta\,\Big|_0^{2\pi}$$

$\boxed{128\pi}$

5) $\displaystyle\int_{-\pi/6}^{7\pi/6} \left[5/2 + 6\sin\theta + (1 - \cos 2\theta) \right] d\theta$ $\qquad \sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$

$$\int_{-\pi/6}^{7\pi/6} \left[7/2 + 6\sin\theta - \cos 2\theta \right] d\theta$$

$$7/2\,\theta - 6\cos\theta - \frac{1}{2}\sin 2\theta \Big|_{-\pi/6}^{7\pi/6}$$

$$\frac{49\pi}{12} - 6\cos 7\pi/6 - \frac{1}{2}\sin\frac{7\pi}{3} - \left(-\frac{7\pi}{12} - 6\cos(-\pi/6) - \frac{1}{2}\sin(-\pi/3)\right)$$

$$\frac{56\pi}{12} - 6\left(-\frac{\sqrt{3}}{2}\right) - \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right) + 6\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{2}\left(-\frac{\sqrt{3}}{2}\right)$$

$$\frac{14\pi}{3} + \frac{6\sqrt{3}}{2} - \frac{\sqrt{3}}{4} + \frac{6\sqrt{3}}{2} - \frac{\sqrt{3}}{4}$$

$$\frac{14\pi}{3} + \frac{22\sqrt{3}}{4}$$

$$\boxed{\frac{14\pi}{3} + \frac{11\sqrt{3}}{2} \approx 24.187}$$