Triple Integrals in Cylindrical Coordinates

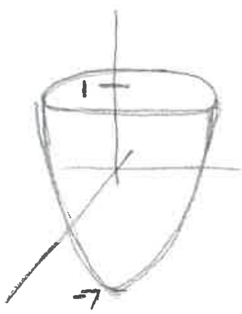
$$\iiint_D f(r,\theta,z)\,dV = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=h_1(\theta)}^{r=h_2(\theta)}\int_{z=g_1(r,\theta)}^{z=g_2(r,\theta)} f(r,\theta,z)\,dz\,r\,dr\,d\theta.$$

$$x = r\cos\theta,\ y = r\sin\theta,\ z = z$$

$$r^2 = x^2 + y^2,\ \tan\theta = y/x.$$

1. Evaluate $\iiint_E 4xy\,dV$ where E is the region bounded by $z = 2x^2 + 2y^2 - 7$ and $z = 1$

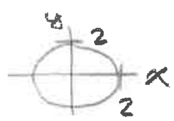
projection on xy plane



$1 = 2x^2 + 2y^2 - 7$

$8 = 2x^2 + 2y^2$

$4 = x^2 + y^2$



$2x^2+2y^2-7 \le z \le 1$

$0 \le \theta \le 2\pi$

$0 \le r \le 2$

$2r^2 - 7 \le z \le 1$

$$\int_0^{2\pi}\int_0^{2}\int_{2r^2-7}^{1} 4r\cos\theta\, r\sin\theta\, r\, dz\,dr\,d\theta$$

$$\int_0^{2\pi}\int_0^{2} 4r^3 z\cos\theta\sin\theta\Big|_{2r^2-7}^{1} dr\,d\theta$$

$$\int_0^{2\pi}\int_0^{2} (32r^3 - 8r^5)\cos\theta\sin\theta\,dr\,d\theta$$

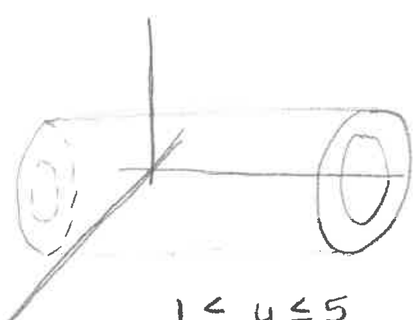
$$\int_0^{2\pi} (8r^4 - 4/3\, r^6)\cos\theta\sin\theta\Big|_0^2 d\theta = \int_0^{2\pi}\frac{128}{3}\cos\theta\sin\theta\,d\theta$$

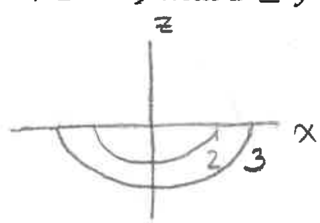
$$= \frac{128}{6}\sin^2\theta\Big|_0^{2\pi}$$

$$= 0$$

2. Evaluate $\iiint_E e^{-x^2-z^2}\,dV$ where E is the region between the two cylinders $x^2 + z^2 = 4$ and $x^2 + z^2 = \cancel{4}\ 9$ with $1 \le y \le 5$ and $z \le 0$

project on xz plane





$1 \le y \le 5$

$\pi \le \theta \le 2\pi$

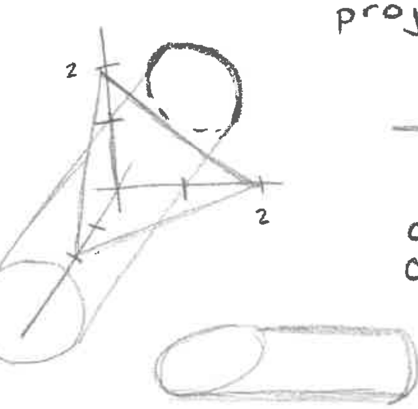
$2 \le r \le 3$

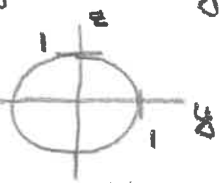
$$\int_{\pi}^{2\pi}\int_2^3\int_1^5 e^{-r^2}\, r\,dy\,dr\,d\theta$$

$$= 0.1143$$

3. Evaluate $\iiint_E z\,dV$ where E is the region bounded by $x + y + z = 2$ and $x = 0$ and inside the cylinder $y^2 + z^2 = 1$

project on yz plane





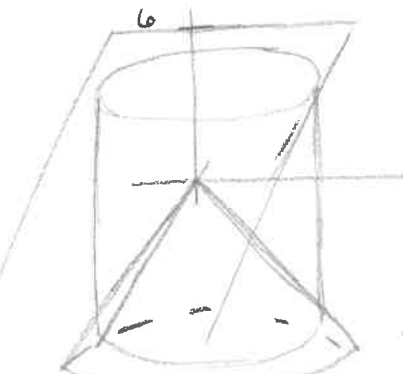
$0 \le r \le 1$
$0 \le \theta \le 2\pi$

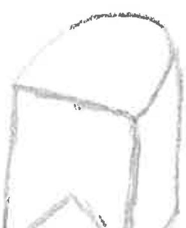
$$\int_0^{2\pi}\int_0^{1}\int_0^{2-r\cos\theta-r\sin\theta} r\cos\theta\, r\, dx\,dr\,d\theta$$

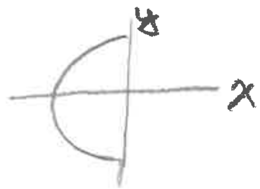
$$= \frac{-\pi}{4}$$

$0 \le x \le 2-y-z = 2-r\sin\theta-r\cos\theta$

4. Use a triple integral to determine the volume of the region below $z = 6 - x$, above $z = -\sqrt{4x^2 + 4y^2}$ inside the cylinder $x^2 + y^2 = 3$ with $x \le 0$







$$\int_{\pi/2}^{3\pi/2}\int_0^{\sqrt{3}}\int_{-2r}^{6-r\cos\theta} r\, dz\,dr\,d\theta$$

$\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$

$= 42.6212$

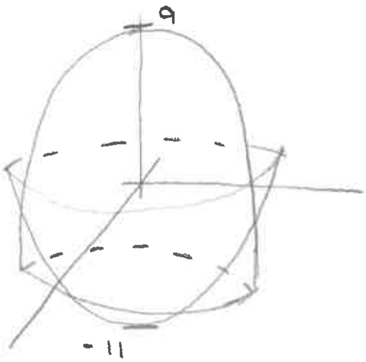
$-\sqrt{4r^2} \le z \le 6-x$ $\quad 0 \le r \le \sqrt{3}$

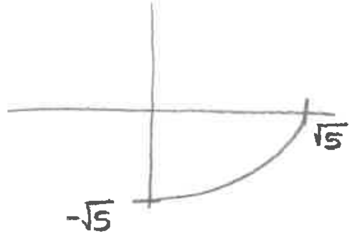
$-2r \le z \le 6 - r\cos\theta$

5. Evaluate the following integral by first converting to an integral in cylindrical coordinates.

$$\int_0^{\sqrt{5}}\int_{-\sqrt{5-x^2}}^{0}\int_{x^2+y^2-11}^{9-3x^2-3y^2} 2x - 3y\, dz\, dy\, dx$$

$$\int_{3\pi/2}^{2\pi}\int_0^{\sqrt{5}}\int_{r^2-11}^{9-3r^2} (2r\cos\theta - 3r\sin\theta)r\, dz\,dr\,d\theta$$





$0 \le r \le \sqrt{5}$

$r^2 - 11 \le z \le 9 - r^2$ $\quad \frac{3\pi}{2} \le \theta \le 2\pi$

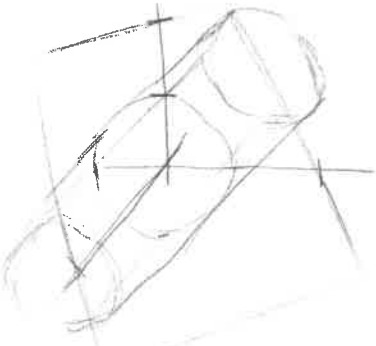
3. Evaluate $\iiint_E z\,dV$ where E is the region bounded by $x + y + z = 2$ and $x = 0$ and inside the cylinder $y^2 + z^2 = 1$

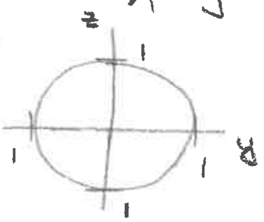
$x = 2 - y - z$ $\quad 0 \le x \le 2 - y - z$

$0 \le x \le 2 - r\sin\theta - r\cos\theta$

integrate in terms of x

project onto yz-plane





$$\int_0^{2\pi}\int_0^1\int_0^{2-r\sin\theta-r\cos\theta} r^2\cos\theta \, dx\,dr\,d\theta$$

$$\int_0^{2\pi}\int_0^1 r^2(2 - r\sin\theta - r\cos\theta)\cos\theta \, dr\,d\theta$$

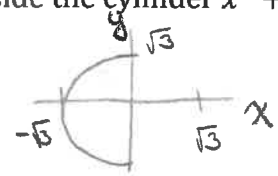
$0 \le r \le 1$

$0 \le \theta \le 2\pi$

$$\int_0^{2\pi} \left(\tfrac{2}{3}r^3 - \tfrac{1}{4}r^4\sin\theta - \tfrac{1}{4}r^4\cos\theta\right)\cos\theta \Big|_0^1 \, d\theta$$

4. Use a triple integral to determine the volume of the region below $z = 6 - x$, above $z = -\sqrt{4x^2 + 4y^2}$ inside the cylinder $x^2 + y^2 = 3$ with $x \le 0$

plane



$\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$

$$\int_{\pi/2}^{3\pi/2}\int_0^{\sqrt{3}}\int_{-2r}^{6-r\cos\theta} r\,dz\,dr\,d\theta$$

$= 42.6212$

$-\sqrt{4r^2} \le z \le 6 - r\cos\theta \quad 0 \le r \le \sqrt{3}$

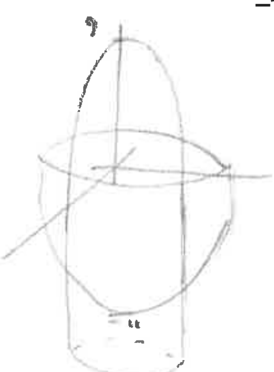
$-2r \le z \le 6 - r\cos\theta$

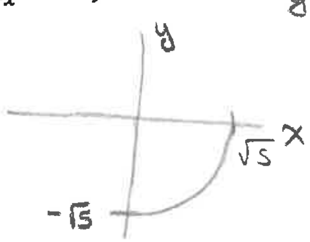
5. Evaluate the following integral by first converting to an integral in cylindrical coordinates.

$$\int_0^{\sqrt{5}}\int_{-\sqrt{5-x^2}}^{0}\int_{x^2+y^2-11}^{9-3x^2-3y^2} 2x - 3y \, dz\,dy\,dx$$

$y = -\sqrt{5 - x^2}$

$y^2 + x^2 = 5$





$$\int_{3\pi/2}^{2\pi}\int_0^{\sqrt{5}}\int_{r^2-11}^{9-3r^2} (2r\cos\theta - 3r\sin\theta)\,r\,dz\,dr\,d\theta$$

$\frac{3\pi}{2} \le \theta \le 2\pi$

$0 \le r \le \sqrt{5}$

$r^2 - 11 \le z \le 9 - 3r^2$

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$$x = r\cos\theta,\ \ y = r\sin\theta,\ \ z = z$$

$$r^2 = x^2 + y^2,\ \ \tan\theta = y/x.$$

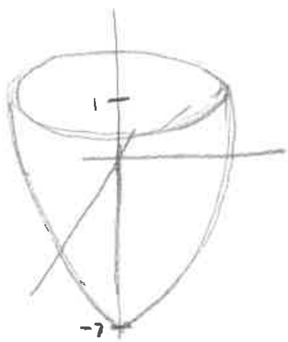
1. Evaluate $\iiint_E 4xy\,dV$ where E is the region bounded by $z = 2x^2 + 2y^2 - 7$ and $z = 1$

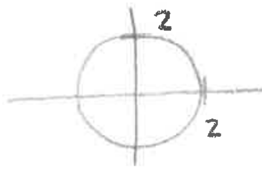
projection on xy plane

$1 = 2x^2 + 2y^2 - 7$

$4 = x^2 + y^2$

$4xy = 4r\cos\theta\, r\sin\theta$





$$\int_0^{2\pi}\int_0^{2}\int_{2r^2-7}^{1} 4r^2\cos\theta\sin\theta\ r\ dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{1} 4r^3 z\cos\theta\sin\theta\Big|_{2r^2-7}^{1} dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{1} (32r^3 - 8r^5)\cos\theta\sin\theta\,dr\,d\theta$$

$2r^2 - 7 \le z \le 1 \quad 0 \le r \le 2$

$0 \le \theta \le 2\pi$

$$= \int_0^{2\pi} (8r^4 - \tfrac{4}{3}r^6)\cos\theta\sin\theta\Big|_0^1 d\theta$$

$$= \int_0^{2\pi} \frac{128}{3}\cos\theta\sin\theta\,d\theta$$

$u = \sin\theta$

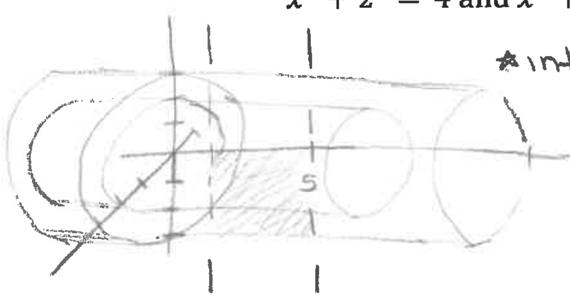
$du = \cos\theta\,d\theta$

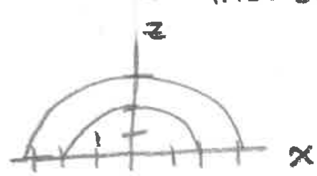
$$= \frac{128}{6}\sin^2\theta\Big|_0^{2\pi}$$

$\boxed{=0}$

2. Evaluate $\iiint_E e^{-x^2-z^2}\,dV$ where E is the region between the two cylinders $x^2 + z^2 = 4$ and $x^2 + z^2 = 9$ with $1 \le y \le 5$ and $z \le 0$

*integrate w/ respect to y instead of z





$1 \le y \le 5$

$2 \le r \le 3$

$\pi \le \theta \le 2\pi$

$e^{-x^2-z^2} = e^{-r^2}$

$$\int_\pi^{2\pi}\int_2^{3}\int_1^{5} re^{-r^2}\,dy\,dr\,d\theta$$

$$\int_\pi^{2\pi}\int_2^{3} re^{-r^2} y\Big|_1^5\,dr\,d\theta$$

$$\int_\pi^{2\pi}\int_2^{3} 4re^{-r^2}\,dr\,d\theta$$

$$\int_\pi^{2\pi} -2e^{-r^2}\Big|_2^3\,d\theta$$

$$\int_\pi^{2\pi} -2e^{-9} + 2e^{-4}\,d\theta$$

$$\theta(-2e^{-9} + 2e^{-4})\Big|_\pi^{2\pi}$$

$$\boxed{= \pi(-2e^{-9} + 2e^{-4})}$$

3) $$\int_0^{2\pi} \left(\frac{2}{3} - \frac{1}{4}\sin\theta - \frac{1}{4}\cos\theta\right)\cos\theta \, d\theta$$

$$\int_0^{2\pi} \frac{2}{3}\cos\theta - \frac{1}{4}\sin\theta\cos\theta - \frac{1}{4}\cos^2\theta \, d\theta$$

$$\int_0^{2\pi} \frac{2}{3}\cos\theta - \frac{1}{8}\sin 2\theta - \frac{1}{8}(1+\cos 2\theta) \, d\theta$$

$$\frac{2}{3}\sin\theta + \frac{1}{16}\cos 2\theta - \frac{1}{8}\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_0^{2\pi}$$

$$\cancel{\frac{2}{3}\sin 2\pi} + \frac{1}{16}\cos 4\pi - \frac{1}{8}\left(2\pi + \cancel{\frac{1}{2}\sin 4\pi}\right) -$$

$$\left(\cancel{\frac{2}{3}\sin 0} + \frac{1}{16}\cos 0 - \frac{1}{8}\left(0 + \cancel{\frac{1}{2}\sin 0}\right)\right)$$

$$\frac{1}{16} - \frac{\pi}{4} - \frac{1}{16}$$

$$\boxed{-\frac{\pi}{4}}$$