Video: https://www.youtube.com/watch?v=wUF-lyyWpUc

Suppose that we want to integrate $f(x,y)$ over the region R. Under the transformation $x = g(u,v)$, $y = h(u,v)$ the region becomes S and the integral becomes,

absolute value of determinant

$$\iint_R f(x,y)\, dA = \iint_S f(g(u,v), h(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \underbrace{d\bar{A}}_{du\,dv}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

*determinant

1. Show that when changing to polar coordinates we have $dA = r\,dr\,d\theta$

* instead of u & v
r & θ

Jacobian = r

$x = r\cos\theta \qquad y = r\sin\theta$

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta - (-r\sin^2\theta)$$

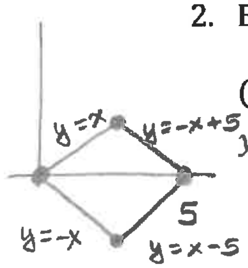
$$= r(\cos^2\theta + \sin^2\theta)$$

$$= r$$

$$dA = \left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| dr\,d\theta = |r|\,dr\,d\theta$$

$$= r\,dr\,d\theta$$

2. Evaluate $\iint_R x + y\,dA$ where R is the trapezoidal region with vertices given by $(0,0)$, $(5,0)$, $(\frac{5}{2},\frac{5}{2})$, and $(\frac{5}{2},-\frac{5}{2})$ using the transformation $x = 2u + 3v$ and $y = 2u - 3v$.



$y = x$

$y = -x$

$y = -x + 5$

$y = x - 5$

when $y = x$:

$2u - 3v = 2u + 3v$

$v = 0$

when $y = -x$

$2u - 3v = -(2u + 3v)$

$u = 0$

when $y = -x + 5$

$2u - 3v = -(2u + 3v) + 5$

$4u = 5$

$u = 5/4$

when $y = x - 5$

$2u - 3v = 2u + 3v - 5$

$-6v = -5$

$v = 5/6$

$0 \le u \le 5/4$

$0 \le v \le 5/6$

$J(u,v) = \begin{vmatrix} 2 & 3 \\ 2 & -3 \end{vmatrix}$

$= -6 - 6 = -12$

$x + y = (2u + 3v) + (2u - 3v)$

$= 4u$

$\int_0^{5/6}\int_0^{5/4} 4u\,|-12|\,du\,dv$

$\int_0^{5/6}\int_0^{5/4} 48u\,du\,dv = \boxed{\frac{125}{4}}$

graph on desmos

3. Evaluate $\iint_R x^2 - xy + y^2\,dA$ where R is the ellipse given by $x^2 - xy + y^2 \le 2$ and using the transformation $x = \sqrt{2}u - \sqrt{\frac{2}{3}}v$, $y = \sqrt{2}u + \sqrt{\frac{2}{3}}v$.

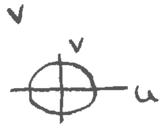
$2 = x^2 - xy + y^2$

$2 = (\sqrt{2}u - \sqrt{2/3}\,v)^2 - (\sqrt{2}u - \sqrt{2/3}\,v)(\sqrt{2}u + \sqrt{2/3}\,v) + (\sqrt{2}u + \sqrt{2/3}\,v)^2$

$2 = 2u^2 - \frac{4}{\sqrt{3}}uv + \frac{2}{3}v^2 - \left(2u^2 - \frac{2}{3}v^2\right) + 2u^2 + \frac{4}{\sqrt{3}}uv + \frac{2}{3}v^2$

$2 = 2u^2 + 2v^2$

$1 = u^2 + v^2$



* can convert to polar

$0 \le r \le 1$

$0 \le \theta \le 2\pi$

$2u^2 + 2v^2 = 2r^2$

$J(u,v) = \begin{vmatrix} \sqrt{2} & -\sqrt{2/3} \\ \sqrt{2} & \sqrt{2/3} \end{vmatrix} = \frac{2}{\sqrt{3}} - -\frac{2}{\sqrt{3}}$

$= \frac{4}{\sqrt{3}}$

* Note now know also:

$x^2 - xy + y^2 = 2u^2 + 2v^2$

$\iint (2u^2 + 2v^2)\,4/\sqrt{3}\,du\,dv$

$\int_0^{2\pi}\int_0^{1} 2r^2\,4/\sqrt{3}\,r\,dr\,d\theta$

$= \boxed{\frac{4\pi}{\sqrt{3}}}$

4. Verify that $dV = \rho^2 \sin \varphi \, d\rho \, d\theta \, d\varphi$ when using spherical coordinates.

$x = \rho \sin \varphi \cos \theta \qquad y = \rho \sin \varphi \sin \theta \qquad z = \rho \cos \varphi$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} = \begin{vmatrix} \sin\varphi \cos\theta & -\rho \sin\varphi \sin\theta & \rho \cos\varphi \cos\theta \\ \sin\varphi \sin\theta & \rho \sin\varphi \cos\theta & \rho \cos\varphi \sin\theta \\ \cos\varphi & 0 & -\rho \sin\varphi \end{vmatrix}$$

$$= \left[-\rho^2 \sin^3\varphi \cos^2\theta + -\rho^2 \sin\varphi \sin^2\theta \cos^2\varphi + 0 \right] -$$

$$\left[\rho^2 \cos^2\varphi \sin\varphi \cos^2\theta + 0 + \rho^2 \sin^3\varphi \sin^2\theta \right]$$

$$= -\rho^2 \sin^3\varphi \left(\cos^2\theta + \sin^2\theta \right) - \rho^2 \sin\varphi \cos^2\varphi \left(\sin^2\theta + \cos^2\theta \right)$$

$$= -\rho^2 \sin^3\varphi - \rho^2 \sin\varphi \cos^2\varphi$$

$$= -\rho^2 \sin\varphi \left(\sin^2\varphi + \cos^2\varphi \right)$$

$$= -\rho^2 \sin\varphi$$

$$dV = \left| -\rho^2 \sin\varphi \right| d\rho \, d\theta \, d\varphi = \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$