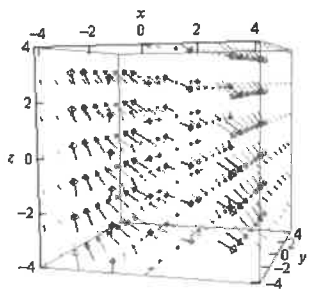
Vector Field in $R^3$ is represented by a vector whose components are functions:

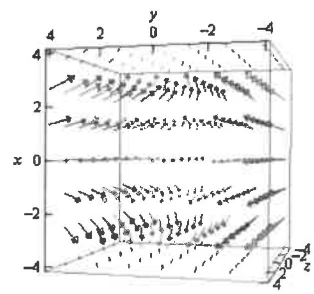
$$\mathbf{F}(x, y, z) = \langle \mathbf{F}_1(x, y, z), \mathbf{F}_2(x, y, z), \mathbf{F}_3(x, y, z) \rangle$$

$$\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$$

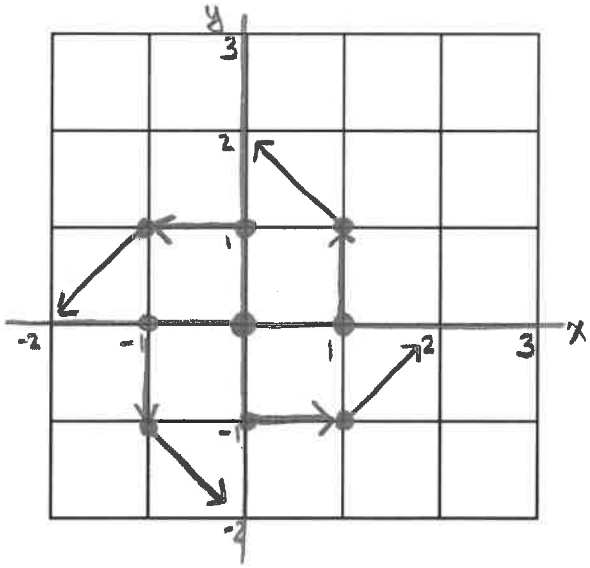
It is not practical to draw a vector field in $R^3$, so we use a computer program to do it for us:

$\vec{F}(x, y, z) = 2x\,\vec{i} - 2y\,\vec{j} - 2x\,\vec{k}$




1. Sketch the vector field for $\vec{F}(x, y) = -y\,\vec{i} + x\,\vec{j}$




$\vec{F}(0, 0) = \langle 0, 0 \rangle$

$\vec{F}(1, 0) = 0\vec{i} + \vec{j} = \langle 0, 1 \rangle$

$\vec{F}(0, 1) = -\vec{i} + 0\vec{j} = \langle -1, 0 \rangle$

$\vec{F}(1, 1) = -\vec{i} + \vec{j} = \langle -1, 1 \rangle$

$\vec{F}(-1, 0) = -\vec{j} = \langle 0, -1 \rangle$

$\vec{F}(-1, 1) = -\vec{i} - \vec{j} = \langle -1, -1 \rangle$

$\vec{F}(1, -1) = \vec{i} + \vec{j} = \langle 1, 1 \rangle$

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

This is a vector field and often called the gradient vector field

2. Find the gradient vector field of:

a. $f(x, y) = x^2 \sin(5y)$

b. $f(x, y, z) = ze^{-xy}$

$f_x = 2x \sin 5y \quad f_y = 5x^2 \cos 5y$

$\nabla f = \langle 2x \sin 5y, 5x^2 \cos 5y \rangle$

$\nabla f = \langle -zye^{-xy}, -zxe^{-xy}, e^{-xy} \rangle$

**Divergence** of a vector field $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$

$$\text{div}(\mathbf{F}) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Often write as a dot product:

$\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$

$$\nabla \bullet \mathbf{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \bullet \langle F_1, F_2, F_3 \rangle = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Properties:

1. div(**F**+**G**)=div(**F**)+div(**G**)

2. div($c$**F**)=$c$div(**F**)

What is divergence?

Example: consider a gas with a velocity vector field given by **F**.

If div(**F**)>0 at a point P, then an outflow of gas occurs, expanding
If div(**F**)<0 at a point P, then the gas is compressing toward P
If div(**F**)=0 the gas is neither compressing nor expanding

3. Evaluate the divergence of $\mathbf{F} = \langle e^{xy}, xy, z^4 \rangle$ at $P = (1, 0, 2)$

$\text{div}(F) = \frac{\partial}{\partial x} e^{xy} + \frac{\partial}{\partial y} xy + \frac{\partial}{\partial z} z^4$

$= ye^{xy} + x + 4z^3$

$\text{div}(F)(1,0,2) = 0e^0 + 1 + 4(2)^3 = \boxed{33}$

**Curl** of a vector field $\mathbf{F}=\langle F_{1,} F_{2,} F_{3} \rangle$

$$\operatorname{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}$$

$$\operatorname{curl}(\mathbf{F}) = \nabla \times \mathbf{F}$$

Properties:

3. curl(**F**+**G**)=curl(**F**)+curl(**G**)

4. curl($c$**F**)=$c$ curl(**F**)

What is curl?

Curl tells us about the rotation.

4. Calculate the curl of $\mathbf{F}=\langle xy, e^{x}, y + z\rangle$

$$\operatorname{curl}(F) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{dz} \\ xy & e^{x} & y+z \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(y+z) - \frac{\partial}{\partial z} e^{x}\right)\vec{i} - \left(\frac{\partial}{\partial x}(y+z) - \frac{\partial}{\partial z} xy\right)\vec{j}$$

$$+ \left(\frac{\partial}{\partial x} e^{x} - \frac{\partial}{\partial y} xy\right)\vec{k}$$

$$= (1-0)\vec{i} - (0-0)\vec{j} + (e^{x} - x)\vec{k}$$

$$= \vec{i} + (e^{x} - x)\vec{k}$$

$$\operatorname{curl}(F) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left( \frac{\partial}{\partial y} F_3 \vec{i} + \frac{\partial}{\partial z} F_1 \vec{j} + \frac{\partial}{\partial x} F_2 \vec{k} \right) -$$

$$\left( \frac{\partial}{\partial y} F_1 \vec{k} + \frac{\partial}{\partial z} F_2 \vec{i} + \frac{\partial}{\partial x} F_3 \vec{j} \right)$$

$$= \left( \frac{\partial}{\partial y} F_3 - \frac{\partial}{\partial z} F_2 \right) \vec{i} + \left( \frac{\partial}{\partial z} F_1 - \frac{\partial}{\partial x} F_3 \right) \vec{j} +$$

$$\left( \frac{\partial}{\partial x} F_2 - \frac{\partial}{\partial y} F_1 \right) \vec{k}$$